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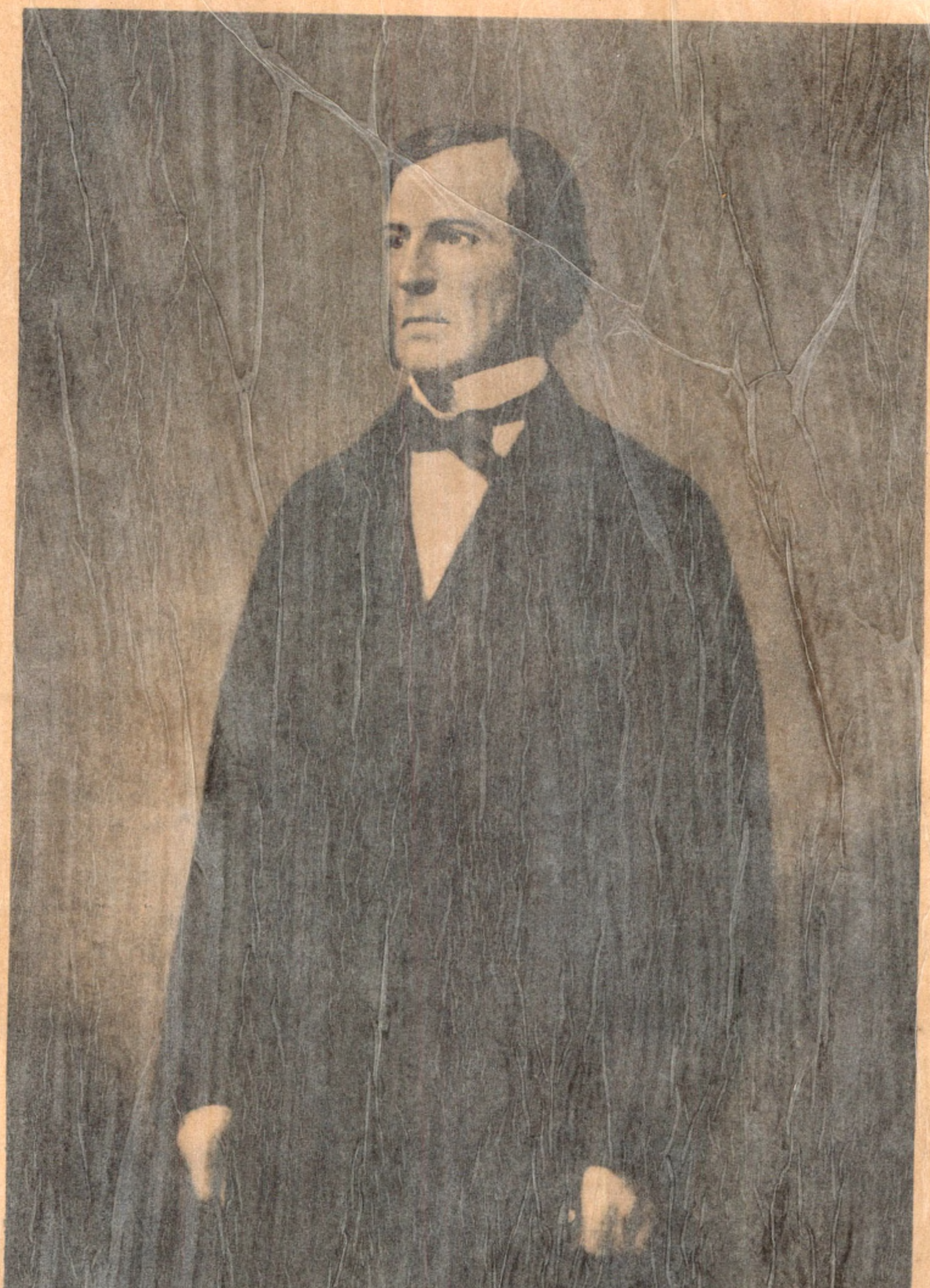
INDEX.

The death of SIMON NEWCOMB, D.Sc., Ph.D., LL.D., D.C.L., etc., on July 11, 1909, brought to a close a life of marvelous activity, productive of many achievements of very high order and value.

Notwithstanding his many other duties, NEWCOMB accepted a call to the Johns Hopkins University as Professor of Mathematics and Astronomy on the departure of Sylvester for England in 1884. At the same time he assumed the editorship of the American Journal of Mathematics.

As Professor and Emeritus Professor in the University, and as Editor-in-Chief and Co-editor of this Journal he exerted here the marked influence of his personality for a quarter of a century



George Boole

The Quadric Spreads Connected with the Configuration $\Gamma^{n+2}_{n+4,n}$, and a Special Case in the Pascal Hexagram.

By W. B. Carver.

In a paper on the Cayley-Veronese Configurations* the author called attention to six conics connected with the configuration $\Gamma^4_{6,2}$. This configuration contains six $\Gamma^3_{5,2}$'s (a $\Gamma^3_{5,2}$ is the well-known Desargues configuration), and with each $\Gamma^3_{5,2}$ is connected a conic ϕ whose polar system sends each point of the configuration into its corresponding line. These six conics lie in a pencil, *i. e.*, they pass through four points. This theorem was proven in the previous paper by a synthetic method which was not capable of immediate extension to the n-dimensional case. The first object of the present paper will be to give an analytic proof for the plane configuration, and then, by a simple extension, to obtain similar theorems for the n-dimensional configuration $\Gamma^{n+2}_{n+4,n}$. In the second part of this paper certain of these pencils of conics connected with the Pascal hexagram will be considered.

I. The Quadric Spreads of the $\Gamma^{n+2}_{n+4,n}$.

1. *The Plane Configuration* $\Gamma^4_{6,2}$. We may take as the equations of any six points in S_4

$$\xi_i = 0 \qquad (i = 1, 2, \ldots, 6)$$

with the relation

$$\sum_i \xi_i = 0$$

Any three of these points, as $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$, determine a plane, 123; and there are twenty such planes. When we take a plane section of this figure

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* *Transactions of the American Mathematical Society*, Vol. VI, pp. 534–545 (October, 1905). The notation used in this present paper was defined in the earlier paper.

we obtain the plane configuration $\Gamma^4_{6,2}$, and the plane ijk of the 4-dimensional figure gives the point ijk* of the configuration $\Gamma^4_{6,2}$. Let the equations of the cutting plane be

$$\begin{cases} \sum_i \alpha_i x_i = 0 \\ \sum_i \beta_i x_i = 0 \end{cases} \qquad (i = 1, 2, \ldots, 5)$$

Then the equations of the twenty points of the $\Gamma^4_{6,2}$ will be

$$\begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \end{vmatrix} = 0 \qquad (i, j, k = 1, 2, \ldots, 6)$$

with the relations

$$\sum_i \xi_i = 0$$

$$\sum_i \alpha_i = 0$$

$$\sum_i \beta_i = 0$$

(The last two relations may be regarded as definitions of α_6 and β_6.)

If we take the coefficients of the ξ's in these equations as co-ordinates of the points we have three superfluous co-ordinates. These superfluous co-ordinates are conducive to symmetry, but they are very inconvenient in determining the equations of the conics ϕ. Hence, sacrificing symmetry to some extent, we eliminate ξ_6 by using the relation

$$\sum_i \xi_i = 0$$

and then simply drop the 4th and 5th co-ordinates, leaving three ordinary homogeneous co-ordinates for each point.† These, together with the co-ordinates dropped, are shown in the table which follows.

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* *Loc. cit.*, p. 535.

† Discarding the 4th and 5th co-ordinates of these points amounts to projecting them from the line 45 upon the plane 123 of the reference figure in S_4.

$$\pi_{ij} \equiv \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}$$

Points.	Co-ordinates.				
123	π_{23}	π_{31}	π_{12}	0	0
456	π_{45}	π_{45}	π_{45}	$\pi_{45} + \pi_{65}$	$\pi_{45} + \pi_{46}$
234	0	π_{34}	π_{42}	π_{23}	0
314	π_{43}	0	π_{14}	π_{31}	0
124	π_{24}	π_{41}	0	π_{12}	0
235	0	π_{35}	π_{52}	0	π_{23}
315	π_{53}	0	π_{15}	0	π_{31}
125	π_{25}	π_{51}	0	0	π_{12}
145	π_{45}	0	0	π_{51}	π_{14}
245	0	π_{45}	0	π_{52}	π_{24}
345	0	0	π_{45}	π_{53}	π_{34}
146	$\pi_{14} + \pi_{64}$	π_{14}	π_{14}	$\pi_{14} + \pi_{16}$	π_{14}
246	π_{24}	$\pi_{24} + \pi_{64}$	π_{24}	$\pi_{24} + \pi_{26}$	π_{24}
346	π_{34}	π_{34}	$\pi_{34} + \pi_{64}$	$\pi_{34} + \pi_{36}$	π_{34}
156	$\pi_{15} + \pi_{65}$	π_{15}	π_{15}	π_{15}	$\pi_{15} + \pi_{16}$
256	π_{25}	$\pi_{25} + \pi_{65}$	π_{25}	π_{25}	$\pi_{25} + \pi_{26}$
356	π_{35}	π_{35}	$\pi_{35} + \pi_{65}$	π_{35}	$\pi_{35} + \pi_{36}$
236	π_{23}	$\pi_{23} + \pi_{63}$	$\pi_{23} + \pi_{26}$	π_{23}	π_{23}
316	$\pi_{31} + \pi_{36}$	π_{31}	$\pi_{31} + \pi_{61}$	π_{31}	π_{31}
126	$\pi_{12} + \pi_{62}$	$\pi_{12} + \pi_{16}$	π_{12}	π_{12}	π_{12}

Now the conic ϕ_i is such that its polar system sends any point of the configuration whose symbol contains the digit i into the line whose symbol contains the digit i together with the three digits not contained in the symbol of the point.* Thus ϕ_1 sends the point 123 into the line 1456, ϕ_6 sends this line 1456 into the point 236, etc. Since the line 1456 contains the points 456, 156, 146, and 145, it is evident that any one of these four points together with the point 123 forms a conjugate pair with respect to ϕ_1. Five independent pairs of conjugate points determine the conic, and its equation may be written at once in terms of the co-ordinates of these points. Having treated 4 and 5 in a special manner, we do not expect the equations of ϕ_4 and ϕ_5 to be symmetrical with

* *Loc. cit.*, p. 543.

those of ϕ_1, ϕ_2, ϕ_3, and ϕ_6. We obtain the following sufficiently simple results:

$$\begin{aligned}
&\phi_4 \equiv \sum_i \pi_{i4}\,(\pi_{i4}+\pi_{54})\,x_i^2 + 2\sum_{ij}\pi_{i4}\,\pi_{j4}\,x_i\,x_j = 0 \\
&\phi_5 \equiv \sum_i \pi_{i5}\,(\pi_{i5}+\pi_{45})\,x_i^2 + 2\sum_{ij}\pi_{i5}\,\pi_{j5}\,x_i\,x_j = 0 \qquad (i, j = 1, 2, 3) \\
&\phi_k \equiv \sum_i \left[\pi_{k4}\,\pi_{i5}\,(\pi_{i5}+\pi_{45}) + \pi_{k5}\,\pi_{i4}\,(\pi_{i4}+\pi_{54})\right] x_i^2 \\
&\qquad + 2\sum_{ij}(\pi_{k4}\,\pi_{i5}\,\pi_{j5}+\pi_{k5}\,\pi_{i4}\,\pi_{j4})\,x_i\,x_j \equiv \pi_{k4}\,\phi_5 + \pi_{k5}\,\phi_4 = 0 \qquad (k = 1, 2, 3, 6)
\end{aligned}$$

The fact that the six conics lie in a pencil is evident at once from these equations.

[The π's may be regarded as co-ordinates in S_4 of the cutting plane, an extension of Plücker's line co-ordinates in ordinary space. There are altogether $\binom{6}{2}$ or 15 of them. Between these 15 π's, however, there are six independent* quadratic relations of the type

$$\pi_{12}\,\pi_{34} + \pi_{13}\,\pi_{42} + \pi_{14}\,\pi_{23} = 0$$

In finding the equations of the ϕ's, these six relations have been used to eliminate six of the π's. Hence these equations are expressed in terms of nine π's, between which there are only the two linear relations

$$\Sigma\,\pi_{i4} = 0$$

and $\qquad\qquad (i = 1, 2, \ldots, 6)$

$$\Sigma\,\pi_{i5} = 0$$

If we use these two relations to eliminate π_{64} and π_{65}, the equations of the ϕ's will contain only seven homogeneous, or six non-homogeneous, independent constants—the proper number to fix a plane in S_4.]

2. *The n-Dimensional Case.* The $\binom{n+4}{3}$ points of the configuration $\Gamma^{n+2}_{n+4,\,n}$ may be given by the equations

$$\begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \end{vmatrix} = 0 \qquad (i, j, k = 1, 2, \ldots, n+4)$$

with the relations

$$\sum_i \xi_i = 0$$
$$\sum_i \alpha_i = 0$$
$$\sum_i \beta_i = 0$$

* There are altogether $\binom{6}{4}$ or 15 such relations. We obtain a set of six which are independent by selecting the six which contain a particular π, as π_{12}.

With this configuration are connected $n+4$ quadric spreads. The spread ϕ_i is defined by the fact that its polar system sends any point whose symbol contains the digit i into the co-point (or S_{n-1}) whose symbol contains the digit i and the $n+1$ digits not contained in the symbol of the point.* The equations of these quadrics are

$$\phi_{n+2} \equiv \sum_i \pi_{i(n+2)} \left(\pi_{i(n+2)} + \pi_{(n+3)\,(n+2)}\right) x_i^2 + 2 \sum_{ij} \pi_{i(n+2)}\, \pi_{j(n+2)}\, x_i\, x_j = 0$$

$$\phi_{n+3} \equiv \sum_i \pi_{i(n+3)} \left(\pi_{i(n+3)} + \pi_{(n+2)\,(n+3)}\right) x_i^2 + 2 \sum_{ij} \pi_{i(n+3)}\, \pi_{j(n+3)}\, x_i\, x_j = 0$$

$$\phi_k \equiv \pi_{k(n+2)}\, \phi_{n+3} + \pi_{k(n+3)}\, \phi_{n+2} = 0$$

$$(i, j = 1, 2, \ldots, n+1 \text{ and } k = 1, 2, \ldots, n+1, n+4)$$

It is evident that these $n+4$ quadrics lie in a pencil, *i. e.*, that they all pass through an $(n-2)$-way spread of the 4th degree.† The equations contain $2n+5$ π's which are connected only by the two linear relations

$$\sum_i \pi_{i(n+2)} = 0$$

$$\sum_i \pi_{i(n+3)} = 0$$

We have then $2n+3$ homogeneous, or $2n+2$ non-homogeneous, independent constants.

3. *The $\Gamma^{n+2}_{n+4,n}$ Determined by the Quadrics.* If to the $2n+2$ constants in our equations of the quadrics we add the n^2+2n constants of a collineation in S_n, we have n^2+4n+2, the proper number of constants for the *general* pencil of $n+4$ quadrics in S_n. The configuration $\Gamma^{n+2}_{n+4,n}$ is also determined by n^2+4n+2 arbitrary constants.‡ This suggests that the pencil of $n+4$ quadrics may be taken arbitrarily, and that the configuration with which they are connected will then be determined. This may be verified analytically.

Consider, for simplicity, the plane case. The polar system of the conic ϕ_1 sends the point 123 into the line 1456, etc. Hence, using the ϕ's as operators, and operating on the point 123 successively with ϕ_1, ϕ_4, ϕ_2, ϕ_5, and ϕ_3, it is sent

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*Cf. Veronese, Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen, etc., *Mathematische Annalen*, Vol. XIX (1882), p. 194.

†The simple one-dimensional case of this theorem is interesting, and is, to the best of the writer's knowledge, new. The $\Gamma^3_{5,1}$ is the configuration of ten points on a straight line obtained by making an arbitrary line-section of the Desargues figure. It is well known that we can pick out of these ten points (in five different ways) a set of six which form three pairs of a quadratic involution. Our theorem gives the additional fact that *the five pairs of double points of these involutions are pairs of another involution.*

‡Author's paper, *loc. cit.*, Theorem IX.

into the line 1236, and this line passes through the point 123. The equations of the ϕ's being known, this gives a quadratic relation which the co-ordinates of the point 123 must satisfy. If now we replace ϕ_4 and ϕ_5 by ϕ_5 and ϕ_6 respectively, the point 123 will be sent into the line 1234; and this gives a second similar relation. These two equations are sufficient to determine the point 123; but since they are quadratic equations, there will be four solutions. Having thus determined the point 123 (or any other point of the configuration) the remainder of the configuration is *uniquely* determined, being readily built up by the *linear* process of taking poles and polars with respect to the ϕ's. A given pencil of six conics may then be connected with any one of *four* $\Gamma^4_{6,2}$ configurations which are determined by the conics.

A similar procedure serves for any $\Gamma^{n+2}_{n+4,n}$. The quadrics $\phi_1, \phi_4, \phi_2, \phi_5$, and ϕ_3 will send the point 123 into the co-point (or S_{n-1}) $12367 \ldots (n+4)$, giving a quadratic relation which must be satisfied by the co-ordinates of the point 123. Replacing ϕ_4 and ϕ_5 by ϕ_5 and ϕ_6 respectively, then ϕ_5 and ϕ_6 by ϕ_6 and $\phi_7, \ldots$ and finally ϕ_{n+2} and ϕ_{n+3} by ϕ_{n+3} and ϕ_{n+4}, we obtain, in all, n equations,* which are sufficient to determine the point 123. Since they are quadratic relations, there will be 2^n solutions. Hence, a given pencil of $n+4$ quadric spreads in S_n may be connected with any one of 2^n $\Gamma^{n+2}_{n+4,n}$ configurations which are determined by the quadrics.

4. *Quadric Spreads Connected with the General Configuration* $\Gamma^{\nu}_{n,r}$. The theorems in the author's previous paper† concerning the conics connected with the general plane configuration $\Gamma^{\nu}_{n,2}$ may now be readily extended. We have first the following:

Connected with every $\Gamma^{\nu}_{n,r}$ (*where* $\nu \overline{>} r+2$ *and* $n \overline{>} \nu + 2$) *are* $\binom{n}{n-r-3}\binom{n-r-3}{\nu-r-1}$ *quadrics lying by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-2}$ *pencils, each quadric in* $\nu - r - 1$ *pencils.*

And combining this theorem with its dual, we have:

Connected with every $\Gamma^{\nu}_{n,r}$ (*where* $\nu \overline{>} r+2$ *and* $n \overline{>} \nu + 3$) *are* $\binom{n}{n-r-3}\binom{n-r-3}{\nu-r-1}$ *quadrics which lie by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-2}$ *pencils, each quadric in* $\nu-r-1$ *pencils; and which also lie by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-1}$ *ranges, each quadric in* $n-\nu-2$ *ranges.*

---

* There are, in fact, $\binom{n+1}{2}$ such conditions on the point 123; but n of them, *which are independent*, may be chosen in the manner indicated.

† *Loc. cit.*, pp. 544, 545.

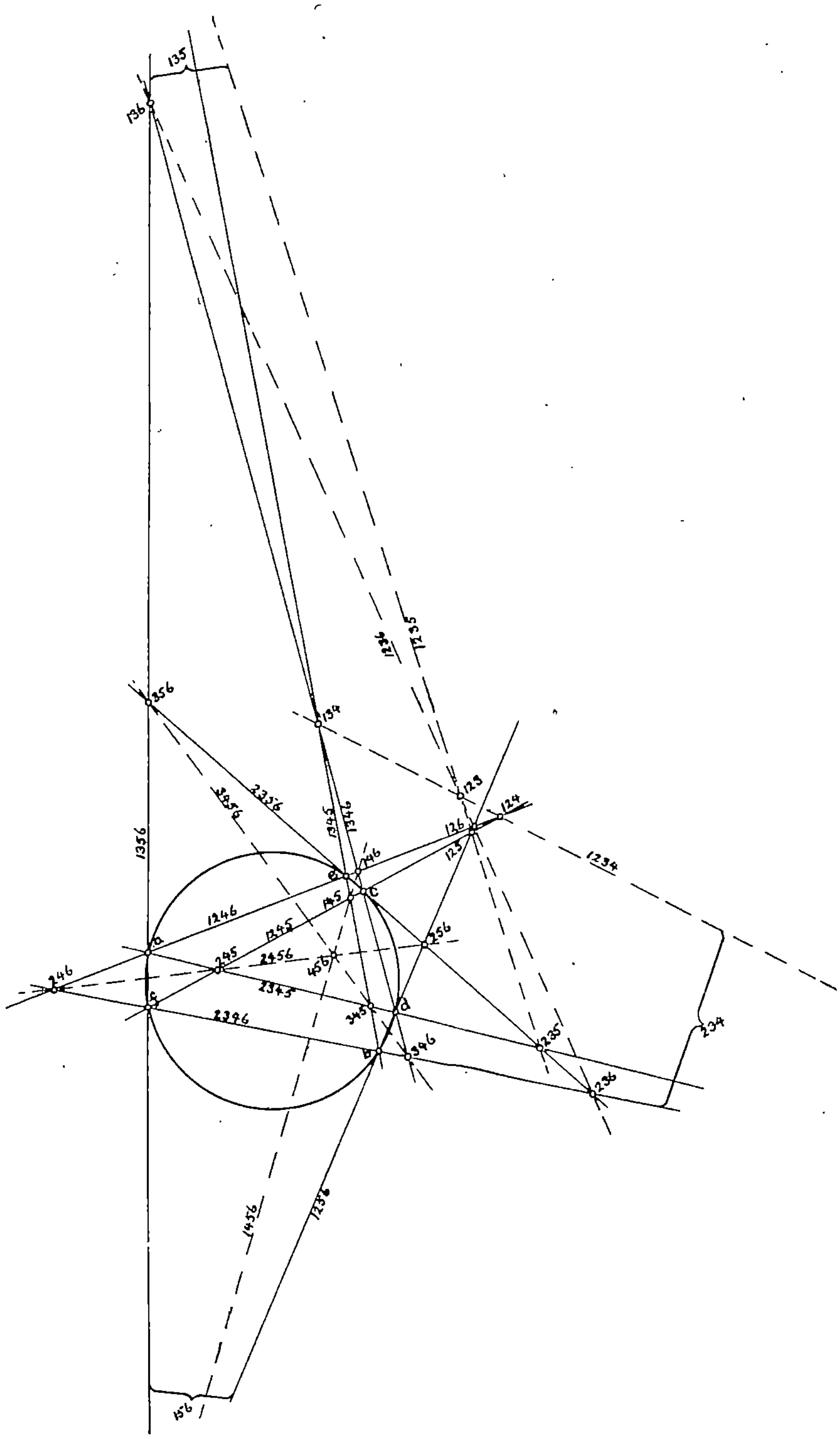
II. Special $\Gamma^4_{6,2}$'s in the Pascal Hexagram.

1. *Cayley's Special* $\Gamma^4_{6,2}$. There are eleven examples of the configuration $\Gamma^4_{6,2}$ in the Pascal hexagram, of which the best known and least interesting is made up of the twenty Steiner points and fifteen Plücker lines.* The remaining ten examples were treated by Cayley.† Let a, b, c, d, e and f be six points on a conic C. Consider the six hexagons obtained by taking a, b and c as alternate vertices and permuting d, e and f in the six possible ways for the other vertices. The six corresponding Pascal lines meet by threes in two Steiner points. These six Pascal lines and the nine lines which make up the sides of the six hexagons are the fifteen lines of a $\Gamma^4_{6,2}$. (See Fig. 1. The Pascal lines are dotted, and the other nine lines are solid.) But these lines meet by threes, not only in the twenty points of the $\Gamma^4_{6,2}$, but also in six extra points—the points on the conic C. The lines and points of the $\Gamma^4_{6,2}$ are designated by the usual combinations of the digits 1, 2, ..., 6. The two Steiner points, being opposite points of the configuration, have the symbols 123 and 456. The three lines 2345, 1356 and 1246, which in the general $\Gamma^4_{6,2}$ would not be collinear, meet in the point a. The two digits not contained in the symbol of each of these three lines are respectively 16, 24 and 35; each pair being composed of one of the digits 1, 2, 3, and one of the digits 4, 5, 6. Paring them off differently, as 15, 26 and 34, we are led to the three lines 2346, 1345 and 1256 which meet in the point b. We may similarly find each of the six sets of three lines which meet at each of the points a, b, ..., f. They are shown in the following table:

	a	b	c	d	e	f
1	6: 2345	5: 2346	4: 2356	6: 2345	4: 2356	5: 2346
2	4: 1356	6: 1345	5: 1346	5: 1346	6: 1345	4: 1356
3	5: 1246	4: 1256	6: 1245	4: 1256	5: 1246	6: 1245

* For a bibliography of the Pascal hexagram and a résumé of the known theorems, see Salmon, *Conic Sections*, p. 379; Richmond, On the Figure of Six Points in Space of Four Dimensions, *Quarterly Journal*, Vol. XXXI (1899), p. 125; Klug, Die Configuration des Pascal'schen Sechseckes (1898).

† Sur Quelques Théorèmes de la Géométrie de Position, *Crelle's Journal*, Vol. XXXI (1846); or *Collected Papers*, Vol. I, p. 317. Also, On Pascal's Theorem, *Quarterly Journal*, Vol. IX (1868); or *Collected Papers*, Vol. VI, p. 129.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236
234
1456
1236
156

FIG. 1.

2. *Analytic Methods.* Since we have the co-ordinates of the points of the general $\Gamma^4_{6,2}$, the equations of its lines may be written, and we may derive at once the conditions that the required sets of lines may be collinear. Thus the lines

$$\begin{array}{lll} & 2345 & x_1 = 0 \\ & 1356 & -\pi_{15}x_1 + (\pi_{15}+\pi_{35}+\pi_{65})x_2 - \pi_{35}x_3 = 0 \\ \text{and} & 1246 & -\pi_{14}x_1 - \pi_{24}x_2 + (\pi_{14}+\pi_{24}+\pi_{64})x_3 = 0 \end{array}$$

are collinear if

$$\begin{vmatrix} 1 & 0 & 0 \\ -\pi_{15} & \pi_{15}+\pi_{35}+\pi_{65} & -\pi_{35} \\ -\pi_{14} & -\pi_{24} & \pi_{14}+\pi_{24}+\pi_{64} \end{vmatrix} = 0$$

By using the relations

$$\sum_i \pi_{i4} = 0, \qquad \sum_i \pi_{i5} = 0 \qquad (i = 1, 2, \ldots, 6)$$

and

$$\pi_{24}\pi_{35} + \pi_{23}\pi_{54} + \pi_{25}\pi_{43} = 0$$

the condition reduces to

$$\pi_{23} + \pi_{43} + \pi_{25} + \pi_{45} = 0$$

The six conditions obtained thus are:

$$\left.\begin{array}{ll} (a) & \pi_{23}+\pi_{43}+\pi_{25}+\pi_{45}=0 \\ (b) & \pi_{31}+\pi_{41}+\pi_{35}+\pi_{45}=0 \\ (c) & \pi_{12}+\pi_{42}+\pi_{15}+\pi_{45}=0 \\ (d) & \pi_{32}+\pi_{42}+\pi_{35}+\pi_{45}=0 \\ (e) & \pi_{13}+\pi_{43}+\pi_{15}+\pi_{45}=0 \\ (f) & \pi_{21}+\pi_{41}+\pi_{25}+\pi_{45}=0 \end{array}\right\} \qquad \text{(I)}$$

As a matter of fact, however, there are only three *independent* conditions upon the π's, the following set of three conditions, for instance, being entirely equivalent* to set (I) above:

$$\left.\begin{array}{ll} (1) & \pi_{41}+\pi_{15}+\pi_{45}=0 \\ (2) & \pi_{42}+\pi_{25}+\pi_{45}=0 \\ (3) & \pi_{43}+\pi_{35}+\pi_{45}=0 \end{array}\right\} \qquad \text{(II)}$$

---

* Any condition of set (I) may be derived from set (II) (and conversely) by using the relations such as

$$\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0$$

and assuming that none of the π's are zero. If one of the π's were zero, we would have a degenerate configuration of no interest.

This set of three conditions is convenient in that they contain only the seven *independent* π's. By adding these three equations, we obtain the useful additional relation

$$(4) \qquad \pi_{46} + \pi_{65} - \pi_{45} = 0$$

When conditions (II) are imposed upon the general $\Gamma^4_{6,2}$, the six sets of three lines will be collinear, *and in addition*, the six meeting points $a, b, \ldots, f$ will lie on a conic. Cayley's special $\Gamma^4_{6,2}$ has, then, eleven degrees of freedom instead of the fourteen degrees of freedom of the general $\Gamma^4_{6,2}$.* The general $\Gamma^3_{5,2}$ (Desargues figure) has also just eleven degrees of freedom; and it is interesting to note that any one of the six $\Gamma^3_{5,2}$'s in our Cayley figure may be taken arbitrarily, and that the remainder of the configuration will then be determined. Consider, for instance, the $\Gamma^3_{5,2}$ made up of those elements containing the digit 6. (See Fig. 1.) The two triangles 146, 246, 346 and 156, 256, 356 are perspective, with the point 456 and the line 1236 respectively as center and axis of perspective. The non-corresponding sides of these two triangles meet in six points of a conic, $a, b, \ldots, f$. The sides of the first triangle join the points $a\,e$, $b\,f$ and $c\,d$; those of the second triangle, $a\,f$, $b\,d$ and $c\,e$. If we pair off these letters in the third way cyclic with these two, $a\,d$, $b\,e$ and $c\,f$, and join these pairs of points, we form a third triangle, 145, 245, 345, perspective with the two original triangles from the same center 456. Adding the two new axes of perspective, 1234 and 1235, our Cayley figure is completed.†

Since the π's are co-ordinates in S_4 of the plane of intersection, the three conditions we have found may be interpreted as geometrical restrictions upon the choice of this plane. ‡ A linear condition on the π's means that the plane belongs to a linear complex of planes. Our plane must then be one of the ∞^3 planes which are common to three such complexes. In set (I), each of the six conditions indicates that the plane is one of the complex of planes which cut a certain line. Condition (a), for example, indicates that the plane cuts the line common to the three spaces 2345, 1356 and 1246.

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* Author's paper, *loc. cit.*, Theorem IX.

† Notice that in this Cayley figure there is symmetry with respect to the two triads of letters a, b, c and d, e, f; also with respect to the two triads of digits 1, 2, 3 and 4, 5, 6. In our analytic treatment, the symmetry with respect to 1, 2, 3 is evident, but the symmetry with respect to 4, 5, 6 has been obscured by the special treatment of these digits.

‡ Cayley treated this $\Gamma^4_{6,2}$ as a $C^3_{6,2}$, *i. e.*, as being the projection upon a plane of the figure of six planes in ordinary space. In his first paper, *loc. cit.*, he stated that for this special $C^3_{6,2}$ there were certain conditions upon the six planes; but in the later paper he corrected this statement, and showed that the six planes might be chosen arbitrarily, and that there were three conditions upon the point chosen as center for the projection.

The plane section may of course be obtained by taking first a space section and then a plane section. In this case, the space section may be taken arbitrarily; and then, to obtain the special Cayley figure, the cutting plane is determined. In the intermediate space configuration $\Gamma^4_{6,8}$ (the figure of two perspective tetrahedra), the combinations 2345, 1356, and 1246 represent planes. Let the point determined by these three planes be called a. Five other points, $b, c, \ldots, f$, may be similarly determined in accordance with the table in paragraph 1. These six points lie in a plane, which we use as the cutting plane. Ten such planes are determined by any $\Gamma^4_{6,8}$ configuration, corresponding to the ten ways of separating the six digits into two triads.

3. *The Conics* ϕ *of the Cayley Figure.* We naturally expect to find some connection between the conic C and the pencil of conics ϕ. Using the relations of set (II), the equations of the conics ϕ_4 and ϕ_5 reduce to

$$\phi_4 \equiv \sum_i \pi_{i4}\,\pi_{i5}\,x_i^2 + 2\sum_{ij} \pi_{i4}\,\pi_{j4}\,x_i\,x_j = 0$$
$$\phi_5 \equiv \sum_i \pi_{i5}\,\pi_{i4}\,x_i^2 + 2\sum_{ij} \pi_{i5}\,\pi_{j5}\,x_i\,x_j = 0$$

The co-ordinates of the six points $a, b, \ldots, f$ are found to be

$$\begin{array}{llll} a: & (0,\ \pi_{53},\ \pi_{24}) & d: & (0,\ \pi_{43},\ \pi_{25}) \\ b: & (\pi_{34},\ 0,\ \pi_{51}) & e: & (\pi_{35},\ 0,\ \pi_{41}) \\ c: & (\pi_{52},\ \pi_{14},\ 0) & f: & (\pi_{42},\ \pi_{15},\ 0) \end{array}$$

and the conic C, through these points, is simply

$$C \equiv 2\sum_i \pi_{i4}\,\pi_{i5}\,x_i^2 + 2\sum_{ij} (\pi_{i4}\,\pi_{j4} + \pi_{i5}\,\pi_{j5})\,x_i\,x_j \equiv \phi_4 + \phi_5 = 0$$

which shows that the conic C lies in the pencil with the conics ϕ. Any conic of this pencil may be expressed in the form

$$\phi_4 + \lambda\,\phi_5 = 0$$

the three conics ϕ_4, ϕ_5 and C becoming thus the *base* conics of the pencil, *i. e.*, the conics having the parameters 0, ∞ and 1 respectively.

If λ_i denote the parameter of the conic ϕ_i, we have

$$\lambda_4 = 0, \quad \lambda_5 = \infty, \quad \text{and } \lambda_i = \frac{\pi_{i4}}{\pi_{i5}} \text{ (when } i = 1, 2, 3 \text{ or } 6)$$

and hence we have the relation

$$\frac{1}{\lambda_1 - 1} + \frac{1}{\lambda_2 - 1} + \frac{1}{\lambda_3 - 1} = \frac{1}{\lambda_4 - 1} + \frac{1}{\lambda_5 - 1} + \frac{1}{\lambda_6 - 1}$$

If the λ's were parameters of points on a line, this equation would indicate that the point whose parameter was 1 had the same second polar with respect to

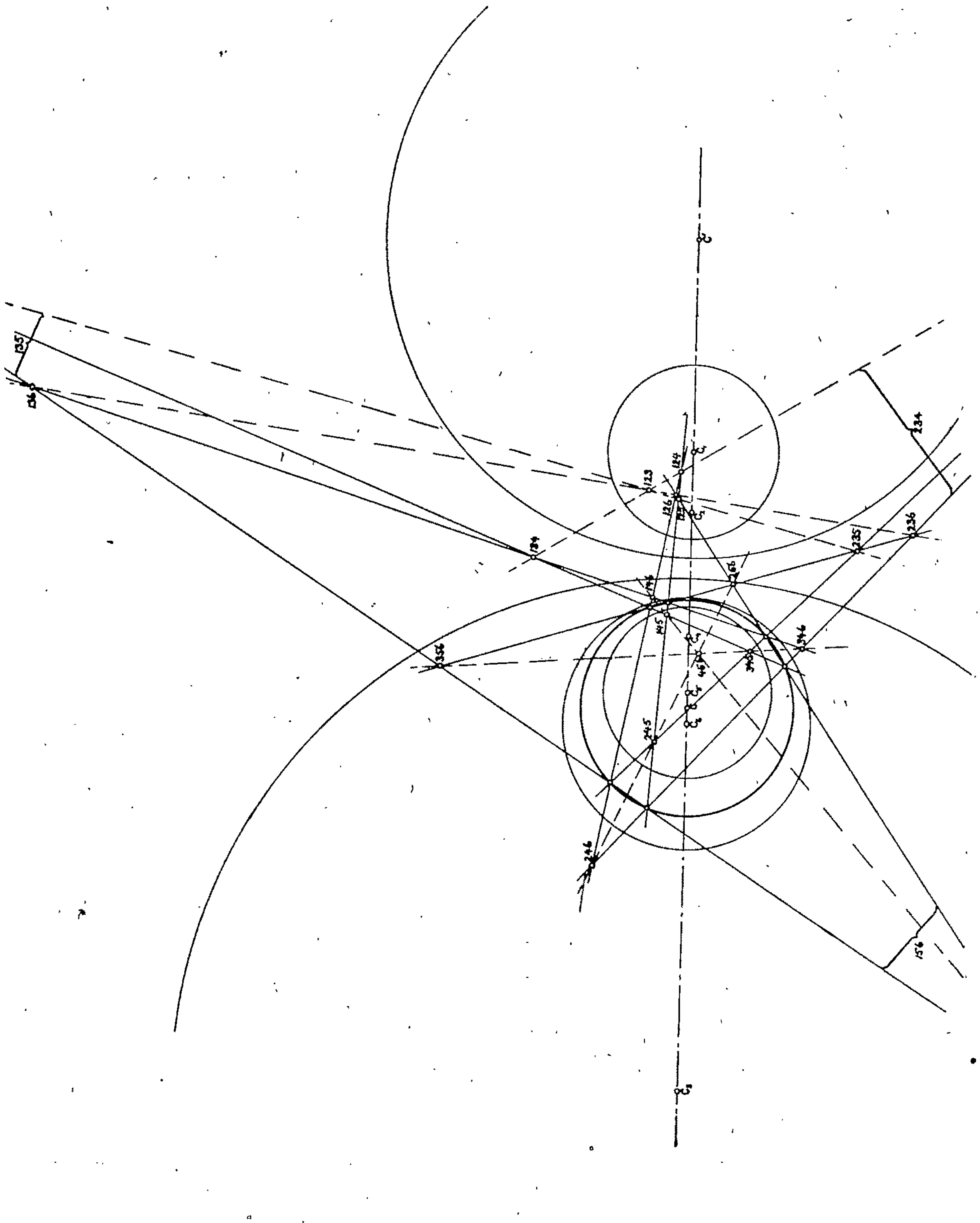

FIG. 2.

The conics C_2 and C_4 are imaginary, but their centers are shown in the figure.

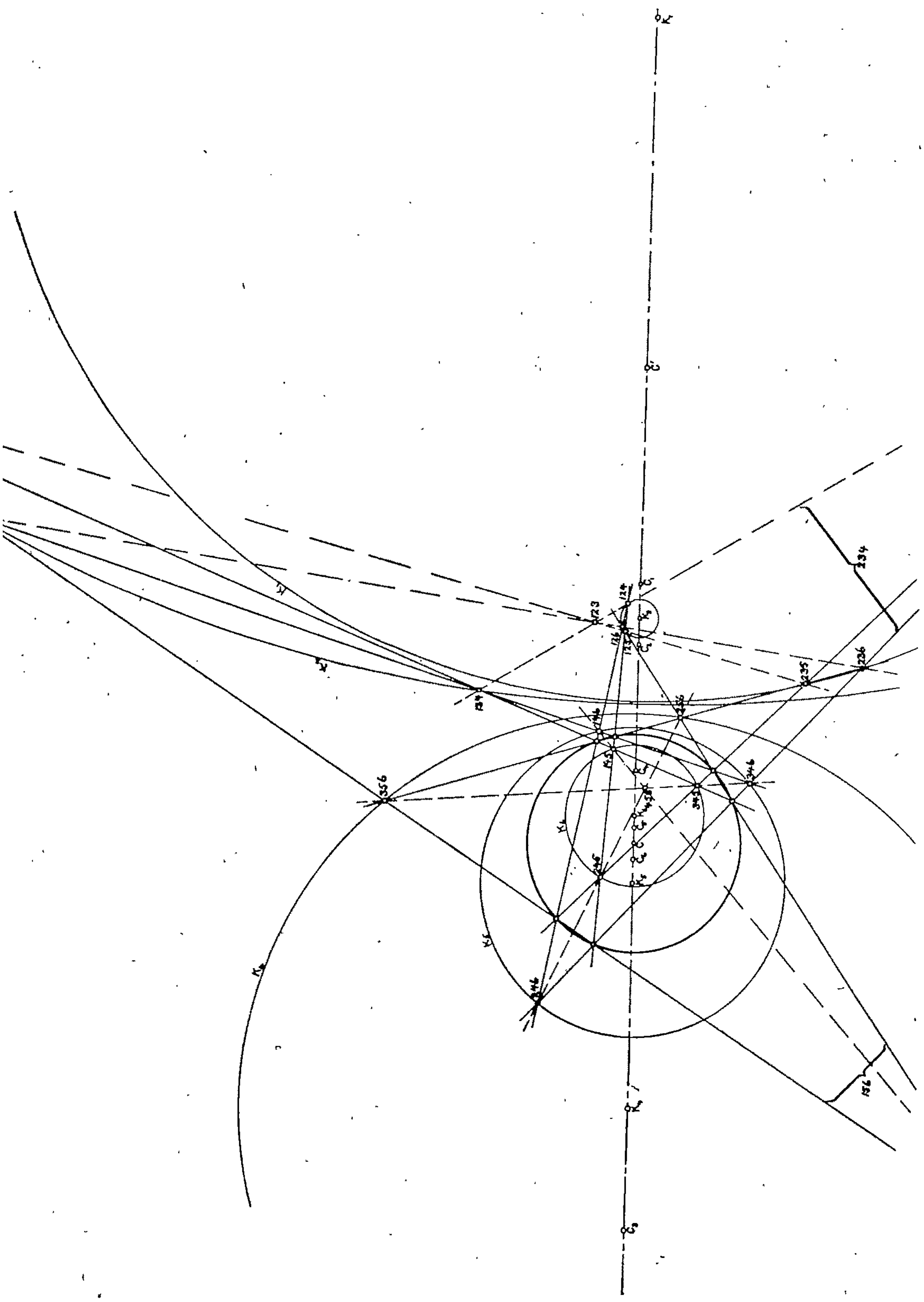

FIG. 3.

the three points λ_1, λ_2, λ_3 that it had with respect to the points λ_4, λ_5, λ_6.* Considering the conics of our pencil, then, as elements, we may say that C has the same second polar with respect to the two triads of conics ϕ_1, ϕ_2, ϕ_3 and ϕ_4, ϕ_5, ϕ_6. The conic C' which is the common second polar of C with respect to the two triads has the parameter (see Fig. 2)

$$\lambda = \frac{\pi_{65} - 3\,\pi_{45}}{\pi_{65} + \pi_{45}}$$

4. *The Conics k.* We next notice that the conic

$$\phi_4 - \phi_5 = 0$$

has no square terms, and is therefore circumscribed to the triangle 145, 245, 345. This is one of the three triangles which are in perspective from the point 456. Because of the peculiar symmetry of our configuration, it follows that there is a conic of the pencil which is circumscribed to each of the six triangles

(1)	234, 235, 236	(4)	156, 256, 356
(2)	134, 135, 136	(5)	146, 246, 346
(3)	124, 125, 126	(6)	145, 245, 345

and these six conics may be designated respectively (see Fig. 3) as

$$k_1,\ k_2,\ k_3,\ k_4,\ k_5 \text{ and } k_6$$

We have seen that the parameter of k_6 is -1. Hence we may say that k_6 is the polar of C with respect to ϕ_4 and ϕ_5. Similarly k_5 must be the polar of C with respect to ϕ_4, ϕ_6; k_1 the polar of C with respect to ϕ_2, ϕ_3; etc. The parameter λ for each of the conics in the pencil is shown in the following table:

Conic	Parameter	Conic	Parameter
C	1	C'	$\dfrac{\pi_{65} - 3\,\pi_{45}}{\pi_{65} + \pi_{45}}$
ϕ_1	$\dfrac{\pi_{14}}{\pi_{15}}$	k_1	$\dfrac{\pi_{24} + \pi_{34}}{\pi_{25} + \pi_{35}}$
ϕ_2	$\dfrac{\pi_{24}}{\pi_{25}}$	k_2	$\dfrac{\pi_{34} + \pi_{14}}{\pi_{35} + \pi_{15}}$
ϕ_3	$\dfrac{\pi_{34}}{\pi_{35}}$	k_3	$\dfrac{\pi_{14} + \pi_{24}}{\pi_{15} + \pi_{25}}$
ϕ_4	0	k_4	$\dfrac{2\,\pi_{64} - \pi_{65}}{\pi_{65}}$
ϕ_5	∞	k_5	$\dfrac{\pi_{64}}{2\,\pi_{65} - \pi_{64}}$
ϕ_6	$\dfrac{\pi_{64}}{\pi_{65}}$	k_6	-1

* These λ's may, for instance, be considered as the parameters of the centers of the conics. These centers have the polar relations as stated (see Fig. 2).

5. *The Common Tangents to Pairs of the Conics k.* It will be seen that eighteen of the twenty points of the configuration — all except the points 123 and 456 — lie on the conics k, three points on each of the six conics. Opposite points of the configuration, such as 124 and 356, are conjugate with respect to each conic of the pencil.* Hence the line joining 124 and 356 is cut by these conics in a quadratic involution of which 124 and 356 are the double points. But since these points lie respectively on k_3 and k_4, they must be the points of contact of a common tangent to these two conics. Similarly, all the points of the configuration except 123 and 456 are points of contact of common tangents to two conics, one from each of the two sets k_1, k_2, k_3 and k_4, k_5, k_6.

If we draw tangents to the conics k_4, k_5 and k_6 respectively at the points 156, 146 and 145, these tangents all touch the conic k_1. The points 156, 146 and 145 lie on the straight line 1456. There must, of course, be some relation between four conics of a pencil in order that such an arrangement be possible. To discover this relation, we may consider the metrically special case of a pencil of circles. Using rectangular Cartesian co-ordinates, and taking the pencil

$$(x^2 + y^2 + 2ax + a^2) + \lambda(x^2 + y^2 - 2ax + a^2) = 0$$

the base circles of the pencil will be the three degenerate circles. By taking the pencil in this way, any *projective* relation between circles of the pencil will be given by a condition on the λ's independent of the constant a, and the result will be applicable to any pencil of conics. The common tangent to the two circles k_1 and k_4 touches k_4 at the point

$$\left(k\frac{\sqrt{\lambda_4}+\sqrt{\lambda_1}}{\sqrt{\lambda_4}-\sqrt{\lambda_1}} \qquad 2k\frac{\sqrt{-\lambda_4(1-\lambda_1)}}{(\sqrt{\lambda_4}-\sqrt{\lambda_1})\sqrt{1-\lambda_4}} \right)$$

and the condition that the three such points for k_4, k_5 and k_6 lie on a straight line is

$$\begin{vmatrix} 1 & \dfrac{1}{\sqrt{\lambda_4}} & \dfrac{1}{\sqrt{1-\lambda_4}} \\ 1 & \dfrac{1}{\sqrt{\lambda_5}} & \dfrac{1}{\sqrt{1-\lambda_5}} \\ 1 & \dfrac{1}{\sqrt{\lambda_6}} & \dfrac{1}{\sqrt{1-\lambda_6}} \end{vmatrix} = 0$$

* Author's paper, *loc. cit.*, p. 544.

When we remove the factors* which simply make $\lambda_4 = \lambda_5$, etc., we have left the condition

$$(\sqrt{\lambda_5} + \sqrt{\lambda_6})\sqrt{1-\lambda_4} + (\sqrt{\lambda_6} + \sqrt{\lambda_4})\sqrt{1-\lambda_5} + (\sqrt{\lambda_4} + \sqrt{\lambda_5})\sqrt{1-\lambda_6} = 0 \qquad \text{(III)}$$

It is noticeable that this condition is entirely free from λ_1. A similar relation must exist, of course, between λ_1, λ_2, and λ_3.

It is interesting to build up the configuration from these conics k. The three degenerate conics of any pencil are chosen; and then the conics k_4, k_5, and k_6 of the pencil are taken with their parameters (referred to the degenerate conics) satisfying condition (III). k_1 may then be taken as any other conic of the pencil. *Any one* of the four common tangents to k_1 and k_4 may be drawn, the points of contact being the points 234 and 156. Then we can select *one* of the common tangents to k_1 and k_5, and *one* of the common tangents to k_1 and k_6, so that the points of contact 146 and 145 on k_5 and k_6 respectively will lie on a line with 156. The sixteen possible selections after the first common tangent is drawn correspond to the sixteen possible choices of sign in condition (III). The remaining points of the configuration will now be linearly determined, together with all the other conics. Although there are thus four different configurations determined when the conics k_1, k_4, k_5 and k_6 are fixed, the conics k_2 and k_3 are the same for each of the four cases, and hence are *uniquely* determined by k_4, k_5, k_6 and k_1. It has been shown (Part I, Section 3) that when the six conics ϕ are given, four different $\Gamma_{6,2}^4$'s are determined by them. If, in this special $\Gamma_{6,2}^4$, the k's are given instead of the ϕ's, the *same* four $\Gamma_{6,2}^4$'s are determined.

If we consider the metrically special case in which the ϕ's and k's are circles, it is evident that when one of the four configurations is drawn, another of the four may be obtained by reflecting the whole figure in the line of centers of the circles. This fact may be readily translated into projective language. Let the conics ϕ and k be given, and let one of the four configurations which they determine be drawn. If a point of the configuration lying upon k_i be projected upon k_i from one of the vertices of the common self-polar triangle of the conics, the point obtained will be the corresponding point of one of the other

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* If we make the substitution $\lambda_i = \sin^2 \alpha_i$, we can readily remove the factors $\sin\frac{\alpha_5-\alpha_6}{2}$, $\sin\frac{\alpha_6-\alpha_4}{2}$ and $\sin\frac{\alpha_4-\alpha_5}{2}$ from the determinant.

configurations. Each vertex of the self-polar triangle may thus be used to obtain the points of one of the configurations. The points a, b, c, d, e, and f on the conic C will be similarly projected into three other sets—a set corresponding to each configuration.

The Pascal hexagram contains ten of these Cayley configurations, each with its pencil of conics. The conic C lies in all of the ten pencils. The conics of any one of these ten pencils determine three new Cayley configurations and three new sets of six points, $a, b, \ldots, f$, on C. The three new sets determined by the conics of one pencil are *not the same* as the three sets determined by the conics of another. It would be interesting to discover just how this extensive system closes in.

Cornell University, *June*, 1907.

The Group-Membership of Singular Matrices.

By Arthur Ranum.

Introduction.

It is well-known* that the totality of non-singular matrices (linear homogeneous substitutions with non-vanishing determinants) form a group under the usual law of composition or multiplication. It has not been so generally recognized that there exist sets of singular matrices (linear substitutions with vanishing determinants), which also form groups† under the same law of composition. For instance, the set of binary matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, $a \neq 0$, obviously form a group of which $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identical matrix.

Not all singular matrices, however, belong to groups; *e. g.*, $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ does not belong to any group whatsoever. It becomes of interest, therefore, to determine whether a given matrix belongs to any group or not; if it does, to find the nature of the largest group to which it belongs; and if it does not belong to any group, to show that there always exists a certain simple relation between it and a uniquely determinable group.

In short, we wish to make a complete classification of all n-ary matrices as to their group-membership and exact relationship to groups. It will also be found that we shall naturally be led to certain incidental results on the roots of a matrix.

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* *Weber's Algebra*, Vol. II, 2nd edition (1899), § 41, p. 168.

† In this paper the word *group* will be used in the generally accepted sense and not in the broader sense recently given to it by Frobenius and Schur in the *Berliner Sitzungsberichte*, 1906, I, p. 209, and by Autonne in the *Comptes Rendus*, 1906, Vol. CXLIII, p. 670. Frobenius and Schur in their other papers employ the word in its usual sense.

PART I. GROUP-MEMBERS.

Definitions and Preliminary Notions.

1. Throughout this paper the elements of the matrices will be unrestricted real or complex numbers, belonging, therefore, to the continuous domain. The number of rows or columns of a matrix will be called its *degree*. As usual, a matrix will be said to be of *rank* r, if it contains at least one r-rowed minor determinant that does not vanish, while all its $r+1$-rowed minor determinants vanish. Following Sylvester, the *vacuity** of a matrix will be defined as the number of zero roots of its characteristic equation. A non-singular matrix of degree n is therefore of vacuity 0 and of rank n, while a singular matrix is of vacuity $v > 0$ and of rank $r < n$. The matrix zero (whose elements are all zero) will be denoted by the symbol 0_n, where n is its degree. A matrix different from zero is said to be *nilpotent*, if one of its powers is equal to zero. The nilpotent matrices are known to be precisely those whose vacuity is equal to their degree and whose rank is > 0.† The unit matrix of degree n will be denoted by the symbol U_n.

Two matrices M and M' are said to be *similar*, if there exists a non-singular matrix L such that $L^{-1}ML = M'$. L is said to *transform* M into M'. In the same way two sets of matrices S and S' are similar, if there exists a non-singular matrix that transforms all the matrices of S into those of S'. A set of matrices S is said to be *completely reducible*, or simply *reducible*,‡ if there exists a similar set S', all of whose matrices are of the form

$$\begin{pmatrix} M_a & 0 \\ 0 & M_b \end{pmatrix},$$

where the symbols M_a and M_b stand for *component* matrices of degrees a and b respectively, and the elements of the remaining rectangular matrices are all zero. Symbolically, we may write

$$S' = \begin{pmatrix} S'_a & 0 \\ 0 & S'_b \end{pmatrix}.$$

* AMERICAN JOURNAL OF MATHEMATICS, Vol. VI (1884), p. 278.

† Frobenius, *Crelle's Journal*, Vol. LXXXIV (1878), p. 15, VI.

‡ It is well-known that there are groups of non-singular matrices that are reducible to the form

$$\begin{pmatrix} G_{aa} & 0 \\ G_{ab} & G_{bb} \end{pmatrix}$$

without being completely reducible in the sense defined above. But it will be shown that every group of singular matrices is reducible in the latter sense.

2. From the theory of elementary divisors* we borrow the theorem that every matrix M' of degree n is similar to a matrix of the form

$$M = \begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \quad (u + v = n), \tag{1}$$

in which the component L_u is non-singular and the component N_v is either nilpotent or zero; v is then the vacuity of M'. If M' is non-singular, $v = 0$ and $M = L_n$; on the other hand, if M' is nilpotent or zero, $u = 0$ and $M = N_n$. Let the characteristic equation of M' be written $\Phi(\lambda) = 0$. It has v roots equal to zero. Let the corresponding elementary divisors of the characteristic determinant of M', which we shall call *vacuous* elementary divisors, be denoted by $\lambda^{e_1}, \ldots, \lambda^{e_s}$, where $e_i \geq e_{i+1}$ $(i = 1, \ldots, s-1)$ and $e_1 + \ldots + e_s = v$. Then $[\mathring{e}_1 \mathring{e}_2 \ldots \mathring{e}_s]$ is the *characteristic*† of the component matrix N_v. Now M can be chosen in such a way that

$$N_v = \begin{pmatrix} E_1 & 0 & \ldots & 0 \\ 0 & E_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & E_s \end{pmatrix}, \tag{2}$$

where

$$E_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \text{ of degree } e_i \quad (i = 1, \ldots, s). \tag{3}$$

When this is done, N_v is said to be in its *canonical form*. When the non-singular component L_u is also chosen in its corresponding canonical form, M is said to be the canonical form of M'. Since the irreducible nilpotent component E_i $(i = 1, \ldots, s)$ is obviously of rank $e_i - 1$, it follows that M, and therefore M', is of rank $r = u + \sum_1^s (e_i - 1) = u + v - s = n - s$. If N_v is irreducible, so that $s = 1$, M' is of rank $n - 1$. On the other hand, if any one of the vacuous elementary divisors of the characteristic determinant of M' is linear, *e. g.*, if

* See Muth's "Elementartheiler" (1899), §§ 77–79, pp. 152–158; also Bôcher's "Introduction to Higher Algebra" (1907), Chap. XXI, and in particular § 100, p. 292. The latter book is especially to be recommended as an introduction both to the theory of matrices and to the theory of elementary divisors.

† See Bôcher's "Higher Algebra," §§ 99 and 100.

$e_s = 1$, then $E_s = 0$, and M has at least one irreducible zero component. If all of the vacuous elementary divisors are linear, then N_v is zero,

$$M = \begin{pmatrix} L_u & 0 \\ 0 & 0_v \end{pmatrix}, \tag{4}$$

and M' is of rank $r = u = n - v$; in other words, its rank is equal to its degree minus its vacuity.

Groups of Singular Matrices.

3. Consider a set of matrices, finite or infinite in number, of the form

$$M = \begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \quad M' = \begin{pmatrix} L_r' & 0 \\ 0 & 0_v \end{pmatrix}, \text{ etc.,}$$

of rank r and vacuity v, in which L_r, L_r', etc., are non-singular, if $r > 0$.* It is apparent that if $L_r L_r' = L_r''$, then $MM' = M''$, and conversely. Therefore the given set of matrices will form a group $G = \begin{pmatrix} G_r & 0 \\ 0 & 0_v \end{pmatrix}$, if, and only if, their non-singular components form a group G_r. In every such group G the identical matrix is

$$I = \begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix} \tag{5}$$

and the inverse of any matrix M is

$$M^{-1} = \begin{pmatrix} L_r^{-1} & 0 \\ 0 & 0_v \end{pmatrix}.$$

In particular, the totality of the matrices of this form constitutes an r^2-parameter continuous group, of which all the other groups are subgroups. It will be noticed that all its matrices have the same rank r.

We wish to show that every possible group of matrices is similar to some group of the kind just described.

The Rank of a Group.

4. LEMMA: *All the matrices of a group have the same rank.*

For if A, of rank a, and B, of rank b, are any two matrices of a group, then matrices C and C^{-1} can be found to satisfy the equations $A = BC$ and $B = AC^{-1}$; and since the rank of the product of two matrices is equal to, or less than, the rank of either factor, the first of these equations shows that $a \leqq b$ and the second shows that $b \leqq a$; therefore $a = b$.

The rank of the matrices of a group will be called the *rank of the group.*

* The extreme case, in which $r = 0$, can be disposed of at once. The only matrix of rank zero is the matrix zero, which clearly constitutes a group of order one.

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5. This lemma compels us to exclude from group-membership every matrix M of the form (1), in which N_v is nilpotent.

For
$$M^\mu = \begin{pmatrix} L_u^\mu & 0 \\ 0 & N_v^\mu \end{pmatrix},$$
and since N_v is nilpotent, μ can be chosen large enough so that $N_v^\mu = 0_v$; in that case M^μ will be of rank u. Now since N_v is different from 0_v, its rank is > 0 and the rank of M is $> u$; therefore M is of higher rank than M^μ. But if M belonged to a group G, M^μ would belong to G and, by the lemma, would be of the same rank as M.

The Conditions of Group-Membership.

6. If a matrix belongs to a group, every similar matrix belongs to a similar group, and if a matrix does not belong to a group, no similar matrix can belong to a group. Now since every matrix is similar to a matrix of the form (1), which we have seen to be a group-member, if the component N_v is zero, and a non-group-member, if N_v is nilpotent, therefore we have proved the

THEOREM: *A necessary and sufficient condition for the group-membership of a matrix (of rank r and vacuity v) is its similarity to a matrix of the form*
$$\begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{6}$$
in which L_r is non-singular.

It is apparent from the proof of this theorem that the different powers of a matrix all have the same vacuity, but not necessarily the same rank; they do all have the same rank, if, and only if, the matrix is a group-member.

7. From this theorem, in view of the principles of elementary divisors and of canonical forms as stated in § 2, we easily derive several *characteristic properties of group-members*, any one of which distinguishes them from all non-group-members. Since all non-singular matrices are group-members, it will be convenient to confine the statement of the results to the only significant case, that in which the matrices are singular.

Thus, *a singular matrix belongs to a group, if, and only if, it satisfies any one of the following equivalent conditions:*

(1) *its powers are all of the same rank;*

(2) *its canonical form has no components that are nilpotent;*

(3) *it is reducible to a non-singular matrix bordered with zeros;*

(4) *the vacuous elementary divisors of its characteristic determinant are all linear;*

(5) *its rank is equal to its degree minus its vacuity.*

If the vacuity of a matrix is zero or one, condition (5) is necessarily satisfied. Therefore every matrix of vacuity one, as well as every non-singular matrix, is a group-member. For instance, every ternary matrix whose characteristic equation has at least two roots different from zero is a group-member.

Idempotent Matrices.

8. Now consider the identical matrix I of any group. It satisfies the condition $I^2 = I$ and is therefore *idempotent*; conversely, every idempotent matrix is the identical matrix of some group. Every such matrix, by the theorem of § 6, is similar to a matrix of the form (6), which in this particular case becomes the matrix (5). Accordingly, the roots of its characteristic equation are equal to one or zero, and not only the vacuous elementary divisors of its characteristic determinant, but all the elementary divisors, are linear. Since the latter condition is also sufficient, we have proved the

THEOREM: *A matrix I (of rank r and vacuity v) is the identical matrix of a group, if, and only if, it satisfies any one of the following equivalent conditions:*

(1) *it is idempotent, that is, $I^2 = I$;*

(2) *it is similar to a matrix of the form*

$$\begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{7}$$

in which U_r is the unit matrix of degree r;

(3) *the roots of its characteristic equation are equal to one or zero, and the elementary divisors of its characteristic determinant are all linear.*

Its rank r is equal to the multiplicity of the root one, while its vacuity v is equal to the multiplicity of the root zero. If $v = 0$, $I = U_n$, while if $r = 0$, $I = 0_n$; in each of these two extreme cases there is only one identical matrix.

Periodic Matrices.

9. Again, suppose that a matrix M belongs to a group of finite order G; then it may be spoken of as *of finite period*, or simply as *periodic*. If m is its period and I is the identical matrix of G, then $M^m = I$ and $M^{m+1} = M$. Conversely, any matrix that satisfies an equation of the form $M^{m+1} = M$ is periodic; for since the powers of M include an identical matrix M^m and an inverse to M, namely M^{m-1}, they form a group.

If, in the theorem of § 6, the matrix (6) is of period m, then $L_r^m = U_r$;

therefore L_r is periodic in the usual non-singular sense, and can be chosen in the canonical form

$$L_r = \begin{pmatrix} \varepsilon_1 \dots 0 \\ \dots\dots \\ 0 \dots \varepsilon_r \end{pmatrix}, \tag{8}$$

in which $\varepsilon_1, \dots, \varepsilon_r$ are mth roots of unity. Thus we have the

THEOREM: *A periodic matrix (of rank r and vacuity v) is characterized by any one of the following equivalent properties:*

(1) *one of its higher powers is equal to its first power;*

(2) *it is similar to a matrix of the form*

$$\begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{6}$$

in which L_r is given by equation (8);

(3) *the roots of its characteristic equation are roots of unity or zero, and the elementary divisors of its characteristic determinant are all linear.*

The Canonical Form of a Group.

10. We now come to the solution of the problem stated in § 3. Let G' be any group whatever, of finite or of infinite order, and let r be its rank, I' its identical matrix, and M' any one of its matrices. Then, by the theorem of § 8, there exists a non-singular matrix L that will transform I' into

$$I = \begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix}. \tag{5}$$

Suppose that L transforms G' into G and M' into

$$M = \begin{pmatrix} L_r & A \\ B & L_v \end{pmatrix},$$

where A and B are rectangular matrices. Since M must satisfy the equations $M = MI = IM$, it follows by direct calculation that $A = 0$, $B = 0$, and $L_v = 0_v$, and that M must be of the form (6); moreover, since its rank is r, its component L_r must be non-singular. But M is obviously any matrix of the group G. Therefore we have arrived at the

THEOREM: *Every group of matrices of rank r is similar to a group*

$$G = \begin{pmatrix} G_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{9}$$

*in which G_r is a group of non-singular matrices of degree r.**

* For the special case in which the group is of finite order, this theorem was proved by Schur, "Neue Begründung der Theorie der Gruppencharaktere," *Berliner Sitzungsberichte*, 1905, I, p. 418, VIII.

G is said to be a *canonical form* of the group. In other words, a set of matrices will form a group, if, and only if, it is similar to a set of matrices of the form (6), in which the non-singular components L_r themselves form a group.

COROLLARY 1: *Every group of singular matrices is reducible to a group of non-singular matrices, each of which is bordered with zeros.*

COROLLARY 2: *Every group of matrices of rank r is simply isomorphic with a group of non-singular matrices of degree r.*

For instance, every group of rank one is simply isomorphic with a unary non-singular group and is therefore Abelian; again, every group of rank two is simply isomorphic with a binary non-singular group.

Entire Groups.

11. Any group of matrices which is not a subgroup of a larger group will be called an *entire group*. An entire group is thus the largest group G to which any one of its matrices M belongs. Every other group to which M belongs is a subgroup of G. Since no two entire groups can have any matrices in common, it follows that every group-member belongs to one and only one entire group. Group-members can therefore be uniquely classified, first according to their ranks, and second according to the entire groups to which they belong.

Now in view of the theorem of the last section, it is evident that all the entire groups of degree n and rank r are similar to one another and that their canonical form is a continuous r^2-parameter group of the form (9). Their number is obviously infinite, unless $r = n$ or $r = 0$. If $r = n$, there is just one entire group, including the non-singular matrices; if $r = 0$, there is one entire group, including the matrix zero alone.

PART II. NON-GROUP-MEMBERS.

The Group-Index of a Matrix.

12. Let us now consider more in detail the properties of matrices that do not belong to any group. By reference to § 5 we see that if any such matrix be raised to a sufficiently high power, the resulting matrix will surely belong to some group. Thus every matrix whatever has some positive integral power* that belongs to a group; and every non-group-member is a root of some group-member.

* A negative or zero power of M obviously has no determinate meaning, unless M is a group-member; in the latter case M^0 is the identical matrix of every group to which M belongs and $M^{-\nu}$ is the inverse of M^ν in every such group.

The exponent of the lowest positive integral power to which a matrix must be raised in order to belong to a group will be called the *group-index* of the matrix. Hence the group-index of every group-member is unity. If M is a non-group-member of group-index μ, then M^μ and all the higher powers of M are group-members (and all belong to the same groups), while the lower powers of M are non-group-members. Therefore, if M^μ is a group-member, while $M^{\mu-1}$ is not, then μ is the group-index of M.

13. By reference to the canonical form of a non-group-member as given by equations (1), (2), (3) of § 2, we see that if μ is its group-index, then the ranks of its successive powers from the first up to the μth form a continually decreasing series of integers,* while the ranks of its higher powers, from the μth on, are all equal.

Moreover, using the notation of § 2, we see that since

$$M^\nu = \begin{pmatrix} L_u^\nu & 0 \\ 0 & N_v^\nu \end{pmatrix},$$

and since N_v^ν is made up of the component matrices E_i^ν $(i = 1, \ldots, s)$, therefore M^ν will be a group-member, if, and only if, $N_v^\nu = 0_v$; that is, if $E_i^\nu = 0_{e_i}$ $(i = 1, \ldots, s)$. Now the lowest value of ν for which the equation $E_i^\nu = 0_{e_i}$ is satisfied is e_i, the degree of E_i. This can be most easily seen by considering a special case. For instance, if $e_i = 4$, then

$$E_i = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^4 = 0_4,$$

and all the higher powers of E_i are equal to 0_4. Therefore the lowest value of ν for which M^ν is a group-member is the largest one of the integers $e_1, \ldots, e_s$, namely e_1. That is, if μ is the group-index of M, then $\mu = e_1$. This result also holds for singular group-members; for in their case $\mu = e_1 = \ldots = e_s = 1$. Thus we have the general

THEOREM: *The group-index of a singular matrix is equal to the degree of that vacuous elementary divisor of its characteristic determinant which is of highest degree.*

It follows that every non-group-member of group-index μ is similar to a matrix having at least one irreducible nilpotent component of degree μ and none of higher degree; its vacuity must be $\geq \mu > 1$.

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* This fact was noticed by Taber, AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 370. His statement of it does not involve the notion of group-membership.

Group-Index and Vacuity.

14. Since matrices exist whose characteristic determinants have any prescribed elementary divisors, therefore the degrees $e_1, \ldots, e_s$ of the vacuous elementary divisors can have any positive integral values such that their sum $\sum_1^s e_i = v \leqq n$, where v is the vacuity and n the degree of the matrix. Hence the group-index of a matrix of vacuity v can have any value from one to v inclusive, but can not be greater than v; that is, its maximum value is v. Similarly, the maximum value of the group-index of a matrix of degree n is n.

Accordingly, the vth powers of all matrices of vacuity v are group-members, and the groups to which they belong are of rank $n-v$. The nth powers of all matrices whatever of degree n are group-members.

If the group-index of a non-group-member M is equal to its vacuity v, then there is just one vacuous elementary divisor $\lambda^v\,(v > 1)$ and M is similar to a matrix having just one irreducible nilpotent component; and conversely.

15. The extreme case in which $v = n$ is that of the nilpotent matrices and the matrix zero. The group-index μ of a nilpotent matrix is evidently the exponent of the lowest power to which it must be raised in order to give the matrix zero. If μ has its maximum value n, then there is only one elementary divisor λ^n and the matrix is irreducible. Conversely, if a singular matrix of degree n is irreducible, it must be nilpotent and its group-index must have the maximum value n.

Group-Index and Rank.

16. Referring again to §2, we see that every non-group-member of degree n, vacuity v, rank r and group-index μ is similar to a canonical matrix having $n-r$ irreducible nilpotent components, the sum of whose degrees is v, while at least one of them is of degree μ and the degrees of all the rest are $\leqq \mu$; moreover, if $v < n$, the canonical matrix has one non-singular component of degree $n-v$, which may, or may not, be further reducible.

The rank r of a singular matrix M of vacuity v can obviously have any positive integral value from $n-v$ (when M is a group-member) up to $n-1$ (when its group-index μ has the maximum value v).

If $r = n - v + 1$, then $e_1 = 2$, $e_2 = \ldots = e_s = 1$, and $\mu = 2$. Hence if the rank of a matrix is one greater than the difference between its degree and its vacuity (so that the latter must be > 1), then its group-index is two.

On the other hand, it is clear that if $\mu = v$, $r = n-1$, and conversely while if $\mu = v-1$, $r = n-2$. That is, every matrix whose group-index is equal to its vacuity is of rank one less than its degree, and conversely; while every matrix whose group-index is one less than its vacuity is of rank two less than its degree.

17. As an illustration of the relations existing between the degree n, the vacuity v, the rank r, and the group-index μ, of a matrix, a few of the corresponding values of these four quantities may be tabulated as follows:

$$
\begin{array}{ll}
v = 0, & r = n, \quad \mu = 1 \\
v = 1, & r = n-1, \quad \mu = 1 \\
v = 2, & \begin{cases} r = n-2, & \mu = 1 \\ r = n-1, & \mu = 2 \end{cases} \\
v = 3, & \begin{cases} r = n-3, & \mu = 1 \\ r = n-2, & \mu = 2 \\ r = n-1, & \mu = 3 \end{cases} \\
v = 4, & \begin{cases} r = n-4, & \mu = 1 \\ r = n-3, & \mu = 2 \\ r = n-2, & \mu = 3 \text{ or } 2 \\ r = n-1, & \mu = 4 \end{cases}
\end{array}
\qquad
\begin{array}{ll}
\multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots} \\
v = n, & \begin{cases} r = 0, & \mu = 1 \\ r = 1, & \mu = 2 \\ r = 2, & \mu = 3 \text{ or } 2 \\ r = 3, & \mu = 4, 3, \text{ or } 2 \ (\text{if } n \geqq 6) \\ \dots\dots\dots\dots & \\ r = n-1, & \mu = n \end{cases}
\end{array}
$$

For the case $v = 4$ the corresponding canonical forms of the nilpotent (or zero) component N_4 are as follows:

$$r = n-4, \quad \mu = 1, \quad N_4 = 0_4$$

$$r = n-3, \quad \mu = 2, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r = n-2, \quad \mu = 3, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r = n-2, \quad \mu = 2, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r = n-1, \quad \mu = 4, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Conditions of Non-Group-Membership.

18. The characteristic properties of a non-group-member M, any one of which distinguishes it from all group-members, may be summarized as follows:

(1) *M^2 is of lower rank than M;*

(2) *M is similar to a matrix having at least one nilpotent component;*

(3) *its characteristic determinant has at least one vacuous elementary divisor that is not linear;*

(4) *its vacuity is >1, and its rank is greater than its degree minus its vacuity;*

(5) *its group-index is >1.*

Non-Group-Members Whose Powers Are not All Distinct.

19. As a special case of a non-group-member, consider a matrix whose powers are not all distinct, while no higher power is equal to the first. Every such matrix M satisfies an equation of the form $M^{\mu+m} = M^{\mu}$ $(\mu > 1)$. If μ and m are the smallest positive integers for which this equation holds, than μ is evidently the group-index of M, while m is the order of the cyclic group formed by the higher powers of M.

For instance, if

$$M = \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ then } M^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and M satisfies the equation $M^6 = M^2$; its group-index is two, and the higher powers M^2, M^3, M^4, M^5 form a cyclic group of order four, whose identical matrix is M^4.

In the canonical form (1) of a matrix of this kind the component L_u must be a periodic non-singular matrix of period m (if $u > 0$) and the component N_v must be a nilpotent matrix of group-index μ. That is, $L_u^m = U_u$ and $N_v^{\mu} = 0_v$. An immedite consequence is the

THEOREM: *A necessary and sufficient condition that the powers of a non-group-member are not all distinct is that the roots of its characteristic equation are either zero or roots of unity (at least two roots being zero) and that at least one of the vacuous elementary divisors of its characteristic determinant is non-linear, while all the non-vacuous elementary divisors are linear.**

* This theorem, without its group implications, is due to Frobenius, "Ueber Lineare Substitutionen und Bilineare Formen," *Crelle's Journal*, Vol. LXXXIV (1878), p. 16, VIII.

Non-Group-Members Whose Higher Powers Are Equal.

20. A still more special case, included under that just considered, is obtained by putting $m=1$; the equation which M satisfies is then $M^{\mu+1}=M^{\mu}\,(\mu>1)$. The cyclic group generated by the higher powers of M is of order one, and M^{μ} is an idempotent matrix. The canonical form of M is

$$\begin{pmatrix} U_u & 0 \\ 0 & N_v \end{pmatrix}.$$

Hence the

THEOREM: *A necessary and sufficient condition that the higher powers of a non-group-member are all equal to one another is that the roots of its characteristic equation are equal to zero or one (at least two roots being zero) and that at least one of the vacuous elementary divisors of its characteristic determinant is non-linear, while all the non-vacuous elementary divisors are linear.*

It is clear that all nilpotent matrices are included in this class; for if M is a nilpotent matrix of degree n and group-index μ, it will satisfy the equation $M^{\mu}=M^{\mu+1}=0_n\,(\mu>1)$.

PART III. THE ROOTS OF A MATRIX.

Associated Matrices.

21. Two matrices M_1 and M_1' will be said to be *associated*, if there exists a non-singular matrix L that transforms them into

$$M=\begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \text{ and } M'=\begin{pmatrix} L_u & 0 \\ 0 & N_v' \end{pmatrix}, \tag{10}$$

respectively, where L_u is non-singular (if $u>0$), and N_v and N_v' are nilpotent or zero. If the vacuity of a matrix is zero or one, it has no associated matrices besides itself. On the other hand, if the vacuity of a matrix is >1, it has an infinite number of associates, which may be said to form a *set of associates.*

In every set of associates just one matrix, the one for which $N_v=0_v$, is a group-member, and all the rest are non-group-members. Thus every group-member of vacuity >1 has an infinite number of associated non-group-members.

Two sets of associates may be called similar, if there exists a non-singular matrix that transforms one into the other; the above set $\{M, M', \ldots\}$ may be called a canonical form of the similar set $\{M_1, M_1', \ldots\}$.

Sets of Associates.

22. THEOREM: *If two sets of associates S and S' have a matrix M' in common, they are identical.*

Proof. In the first place S and S' must obviously be similar sets; there will be no loss of generality in considering one of them, S, to be in its canonical form. The non-singular matrix L that transforms S' into S will transform the common matrix M' into some matrix M of S. Suppose M and M' to be given by equations (10) and let

$$L = \begin{pmatrix} A_u & B \\ C & D_v \end{pmatrix}.$$

Then since $M' L = L M$, we see that L must satisfy the equations

$$\text{(a)}\ L_u A_u = A_u L_u, \qquad \text{(b)}\ L_u B = B N_v,$$
$$\text{(c)}\ N'_v C = C L_u, \qquad \text{(d)}\ N'_v D_v = D_v N_v.$$

From (b) and (c), in view of a theorem due to Frobenius,* we see that $B = 0$, and $C = 0$, and therefore that

$$L = \begin{pmatrix} A_u & 0 \\ 0 & D_v \end{pmatrix};$$

from (a) we see that A_u is commutative with L_u, and therefore that L (and also L^{-1}) will transform the set S into itself. But L^{-1} transforms S back into S'; consequently the two sets S and S' are identical.

This theorem shows that every matrix belongs to one, and only one, set of associates, and that every non-group-member is associated with a uniquely determined group-member.

Pseudogroups.

23. The matrices of an entire group of vacuity > 1, together with all the non-group-members associated with them, constitute a set of matrices, which we shall call the *pseudogroup* associated with the given entire group. Hence a pseudogroup is made up of sets of associates. Since two different sets of associates can not have a matrix in common, and since the same is true of two different entire groups, therefore, similarly, two different pseudogroups can not have a matrix in common. In other words, pseudogroups are mutally exclusive sets of matrices. Accordingly, every matrix of vacuity > 1 belongs to one, and only one, pseudogroup, just as every matrix of vacuity one or zero belongs to one, and only one, entire group.

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* *Crelle's Journal*, Vol. LXXXIV (1878), p. 28, XI. The theorem is stated only for square matrices (bilinear forms), but can be immediately extended to cover the case in which B and C, as here, are rectangular matrices.

Since all entire groups of the same vacuity are similar, therefore all pseudogroups of the same vacuity are similar. The canonical form of every pseudogroup of vacuity v and degree $u+v$ may be written

$$P=\begin{pmatrix} G_u & 0 \\ 0 & H_v \end{pmatrix}, \tag{11}$$

where G_u (if $u>0$) denotes the group of non-singular matrices of degree u, and H_v denotes the totality of nilpotent matrices of degree v plus the matrix 0_v. The nilpotent matrices of degree n are all associated with the matrix zero, and with it form the only pseudogroup of vacuity n.

Since the group G_u, of (11), involves u^2 parameters, while in the set H_v the v^2 elements of each matrix are connected only by v independent relations derived from the vanishing of the roots of its characteristic equation, therefore *every pseudogroup of degree n and vacuity v is a continuous set of matrices depending on $(n-v)^2+v^2-v$ parameters.*

24. From the definition of a pseudogroup P it is clear that if a matrix M belongs to P, every power of M will belong to P; again, since pseudogroups are mutually exclusive, every root of M will belong to P. Hence the

THEOREM: *Every power and every root of a matrix of vacuity >1 belongs to the pseudogroup to which the matrix belongs; similarly, every power and every root of a matrix of vacuity one or zero belongs to the entire group to which the matrix belongs.*

On the other hand, the product of two matrices of a pseudogroup does not necessarily belong to the pseudogroup, because the product of two nilpotent matrices is not necessarily nilpotent.

The Roots of a Group-Member.

25. Let us now consider the distribution of the powers and roots of a matrix within the pseudogroup to which the matrix belongs. In the first place, if a pseudogroup P is of vacuity v, then the vth power of every one of its matrices belongs to the entire group with which P is associated.

Moreover, if, in formula (10) of § 21, the group-indices of M and M' are $\leqq \nu$, then

$$M^\nu=(M')^\nu=\begin{pmatrix} L_u^\nu & 0 \\ 0 & 0_v \end{pmatrix},$$

which we may call L. Accordingly, if we select from a set of associates those matrices whose group-indices are $\leqq \nu$, their νth powers must all be equal to one another and to the νth power, L, of their associated group-member. Therefore,

if M is a νth root of a group-member L, then, among the matrices associated with M, those whose group-indices are $\leq \nu$ are all νth roots of L.

In particular, if $\nu \geqq v$, then absolutely all the matrices associated with M are νth roots of L. That is, *the matrices of any set of associates of vacuity v are all νth roots of one and the same group-member, if $\nu \geqq v$.*

26. To find all the νth roots of any given group-member L of vacuity v, we can therefore proceed as follows: First find all the νth roots of L that are contained within the entire group G to which L belongs. Such roots exist for all positive integral values of the index ν. If v is one or zero, these are the only νth roots of L. But if $v > 1$, and if M is any νth root belonging to G, we then select those of its associated non-group-members, whose group-indices are $\leqq \nu$; in particular, if $\nu \geq v$, we select all its associated non-group-members. The matrices so found will include all the νth roots of L and no other matrices.

It is evident, therefore, that every group-member L possesses roots of any given index ν.

The Roots of a Non-Group-Member.

27. Finally, let M' be a non-group-member of vacuity v and group-index μ belonging to the canonical pseudogroup P of (11), § 23, and let M be any νth root of M'; M will also belong to P. If we put

$$M = \begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \text{ and } M' = \begin{pmatrix} L'_u & 0 \\ 0 & N'_v \end{pmatrix},$$

then M must satisfy the conditions $L_u^\nu = L'_u$ and $N_v^\nu = N'_v$. Now it is not always possible to find a nilpotent matrix N_v to satisfy the latter condition. But if it is possible, so that M' possesses a νth root M, then all its νth roots are connected as follows: Let L and L' be the group-members associated with M and M', respectively; then L is obviously a νth root of L'. Conversely, if L is a νth root of L', M is a νth root of M'. Hence the

THEOREM: *If M' is a non-group-member and L' its associated group-member, then if a νth root of M' exists, every νth root of M' is associated with a νth root of L' and every νth root of L' is associated with a νth root of M'.*

28. In order that M', as defined above, may possess a νth root, the inequality

$$(\mu - 1)\nu < v \tag{12}$$

must be satisfied. For since μ is the group-index of M', the nilpotent matrix N'_v must satisfy the condition $(N'_v)^{\mu-1} \neq 0_v$; hence N_v must satisfy the condition

$(N_v)^{(\mu-1)\nu} \neq 0_v$. This means that the group-index of M is $> (\mu - 1)\nu$. But the maximum value of the group-index of M is its vacuity v, so that $v > (\mu - 1)\nu$. Therefore,

A non-group-member M' of vacuity v and group-index μ does not possess a νth root, unless $(\mu - 1)\nu < v$. In other words, if $(\mu - 1)\nu \geqq v$, M' has no νth root, and, *a fortiori*, it has no root of index $> \nu$.

If in (12) v is fixed and μ has its minimum value 2, it follows that the maximum value of ν is $< v$. Hence, *a non-group-member can not have any roots of index equal to, or greater than, its vacuity.*

If in (12) v is fixed and ν has its minimum value 2, it follows that the maximum value of μ is $< \frac{v}{2} + 1$. Hence, *a non-group-member of vacuity v and group-index $\geqq \frac{v}{2} + 1$ has no square roots, and so has no roots whatever.*

Example. Consider ternary nilpotent matrices ($n = v = 3$), which have the two canonical forms:

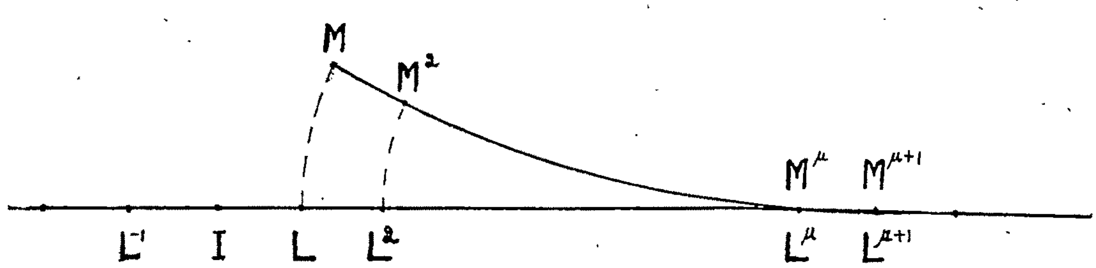
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\mu = 2), \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} (\mu = 3).$$

A has square roots, *e. g.* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, but no roots of higher index than two; on the other hand, B has no roots whatever.

PART IV. ILLUSTRATIONS AND APPLICATIONS.

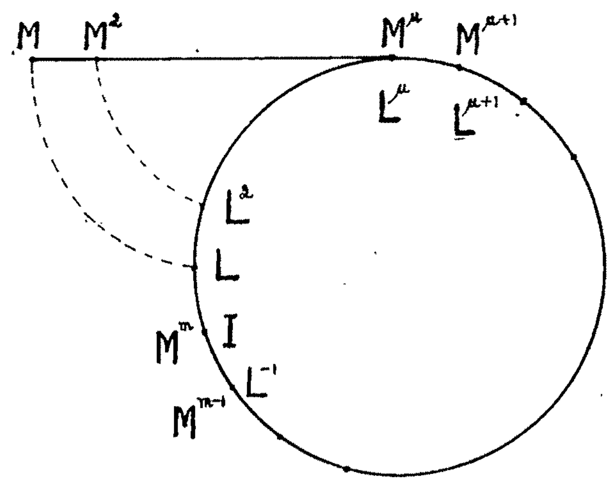
Graphical Representation.

29. In order to lend concreteness to the relation between the powers of a non-group-member M, of group-index μ, and the powers of its associated group-member L, we may represent them by points on two converging lines, thus:



The points on the straight line represent the matrices of the group G generated by L, in which I is the identical matrix and L^{-1} is the inverse of L. M is associated with L, M^2 with L^2, etc.; and since μ is the group-index of M, $M^\mu = L^\mu$ and all the higher powers of M are equal to the corresponding powers of L.* It is plain that although the powers of M from the μth on generate the group G, M^μ itself only generates a subgroup of G.

30. For the special case in which M satisfies the equation $M^{\mu+m} = M^\mu$, the matrices of the group G may be more conveniently represented by points on a circle, thus:



For the sake of simplicity m is here taken $> \mu$. As before, M is associated with L, M^2 with L^2, etc., and $M^\mu = L^\mu$, $M^{\mu+1} = L^{\mu+1}$, etc. Since L is of period m, the identical matrix of G is $I = L^m = M^m$. Moreover, $M^{\mu+m} = L^{\mu+m} = L^\mu = M^\mu$. The powers of M repeat themselves, but never reproduce the first power. M^μ will generate the whole group G, if, and only if, μ is prime to m. An instance in which it generates a subgroup is the example given in § 19.

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* By purely abstract considerations it is easy to show that if M is any mathematical entity whose μth and $(\mu+1)$st powers belong to a group G, then all its higher powers belong to G, provided the powers of M combine under the associative law. For since $(MM)M = M(MM)$, therefore M^μ and $M^{\mu+1}$ are commutative; and since the latter matrices belong to G, $(M^\mu)^{-1}$ belongs to G and is commutative with $M^{\mu+1}$. Let $L = M^{\mu+1}(M^\mu)^{-1} = (M^\mu)^{-1} M^{\mu+1}$; then L belongs to G. Moreover, $L = M \cdot M^\mu (M^\mu)^{-1} = (M_\mu)^{-1} M^\mu \cdot M$. Therefore, if I is the identical element of G, $L = MI = IM$, and M is commutative with I. Hence $L^\lambda = M^\lambda I^\lambda = M^\lambda I$. Now suppose $\lambda = \mu + \nu$, where $\nu > 0$; then $L^\lambda = M^\nu \cdot M^\mu I = M^\nu \cdot M^\mu$ (since M^μ belongs to G) $= M^\lambda$; and since L belongs to G, M^λ belongs to G, when $\lambda \geqq \mu$. That is, every power of M, whose exponent is $\geqq \mu$, belongs to G.

Illustration from Differential Calculus.

31. The phenomenon of a mathematical entity not belonging to any group, while some of its powers do belong to a group, is not peculiar to the theory of matrices; it can also be exemplified elsewhere. For instance, in differential calculus, let D and E denote the operations of differentiating $y = e^{ax} + x^{\mu-1}$ and $z = e^{ax}$, respectively, as to x. Then

$$\begin{aligned} Dy &= a e^{ax} + (\mu - 1) x^{\mu-2} \text{ and } Ez = a e^{ax}, \\ &\dots\dots\dots\dots\dots\dots, \\ D^{\mu} y &= a^{\mu} e^{ax} = E^{\mu} z, \\ &\dots\dots\dots\dots, \\ D^{\nu} y &= a^{\nu} e^{ax} = E^{\nu} z \qquad (\nu \geqq \mu). \end{aligned}$$

Now the powers of E form a group G, provided the inverse operation E^{-1} be defined by the equation $E^{-1} z = \frac{1}{a} e^{ax}$. On the other hand, although the powers of D evidently do not form a group, the higher powers of D, from the μth on, belong to the group G.

As a special case let a be a primitive mth root of unity; then G becomes a cyclic group of order m, and D an operation satisfying the equation

$$D^{\mu+m} y \,(= a^{\mu+m} e^{ax} = a^{\mu} e^{ax}) = D^{\mu} y.$$

Groups of Singular Collineations.

32. By the use of homogeneous point coordinates every matrix of degree n can be written as a linear substitution and interpreted geometrically as a collineation of a linear space, R_{n-1}, of $n-1$ dimensions. Every singular matrix will then carry the points of R_{n-1} into the points of an included space R_{r-1} $(r < n)$. Hence it is clear that in a group of singular collineations the identical collineation will not leave every point unchanged, as is the case in a group of non-singular collineations, and the inverse of a collineation that carries the point P into P' will not necessarily carry P' back into P.

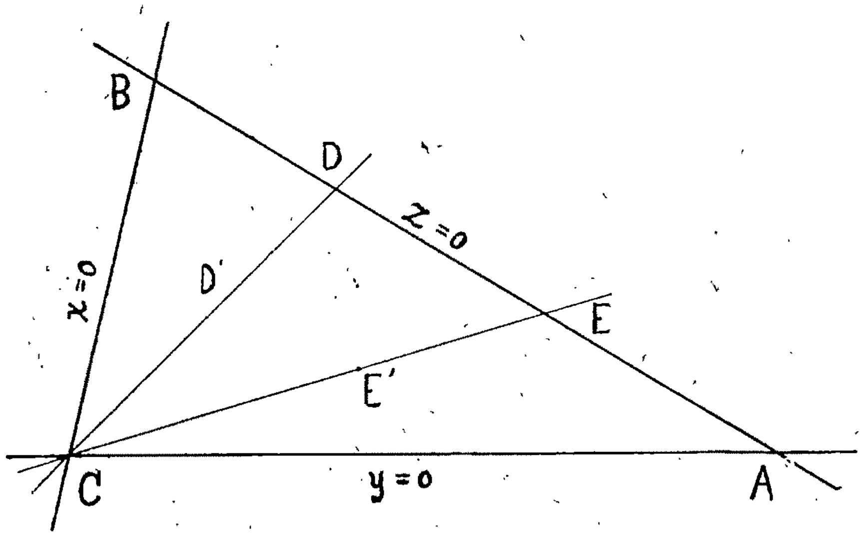
Nevertheless the group concept, as applied to singular collineations, can be justified geometrically, as will be apparent from the consideration of a simple case. Let n be 3 and consider a group generated by the singular matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2 and of infinite period. Then M, $M^0 = I$, and M^{-1} may be written as linear substitutions, thus:

$$M: \begin{cases} x' = x + y \\ y' = y \\ z' = 0 \end{cases} \qquad I: \begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases} \qquad M^{-1}: \begin{cases} x' = x - y \\ y' = y \\ z' = 0 \end{cases}$$

Considered as collineations in the plane they carry all the points of the plane except C $(x = y = 0)$ into points of the line AB $(z = 0)$.



All the points except C of any line passing through C are carried into one and the same point of AB, and the points of AB are transformed projectively among themselves. Let CD be any line passing through C, and suppose that M carries the points of CD into E; then M^{-1} will carry the points of CE into D. I evidently carries the points of CD into D, those of CE into E, etc.

Now the equations $M = IM = MI$ and $MM^{-1} = M^{-1}M = I$, which determine the group-membership of M, are satisfied geometrically. For I carries any point D' of the line CD into D, and M carries D into E; therefore IM carries D' into E, exactly as M does. Similarly MI transforms the points of the plane exactly as M does. Again, M^{-1} carries E' into D, and M carries D into E; therefore $M^{-1}M$ carries E' into E, just as I does. Similarly, MM^{-1} transforms the points of the plane just as I does.

This interpretation can obviously be extended to the general case of any group whatever of singular collineations.

Classification of Binary and Ternary Matrices.

33. An application of the principles derived in this paper will now be made to the classification of binary and ternary matrices, not only with respect to their group-membership, but also with respect to their vacuity v, rank r, group-index μ, and the entire groups and pseudogroups to which they belong. In these simple cases ($n = 2, 3$) the values of v and r completely determine the vacuous elementary divisors of the characteristic determinants of the matrices, and therefore also their characteristics,* so far as the vacuous elementary divisors alone are concerned.

34. *Binary Matrices.* Every binary matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ may be considered as belonging to one of four great classes, as follows:

$$\text{(a)}\quad v = 0,\ r = 2,\ \mu = 1;\ \alpha\delta - \beta\gamma \neq 0;$$

these matrices, ∞^4 in number, non-singular, form a single entire group.

$$\text{(b)}\quad v = 1,\ r = 1,\ \mu = \mathring{1};\ \alpha\delta - \beta\gamma = 0,\ \alpha + \delta \neq 0;$$

∞^3 matrices; characteristic $[1\,\mathring{1}]$; canonical form $\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix}$, $\alpha \neq 0$; they form ∞^2 similar entire groups, each containing ∞^1 matrices.

$$\text{(c)}\quad v = 2,\ r = 1,\ \mu = 2;\ \alpha\delta - \beta\gamma = \alpha + \delta = 0,$$

at least one element $\neq 0$; ∞^2 matrices; characteristic $[\mathring{2}]$; canonical form $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$; they are nilpotent, all similar, and their square is zero; they are all associated with zero and with it form a single pseudogroup.

$$\text{(d)}\quad v = 2,\ r = 0,\ \mu = 1;\ \alpha = \beta = \gamma = \delta = 0;$$

characteristic $[(\mathring{1}\,\mathring{1})]$; one matrix, zero, forming an entire group of order one.

The matrices of classes (a), (b), and (d) are group-members, and those of class (c) are non-group-members. Thus all binary non-group-members are nilpotent. There are three kinds of binary idempotent matrices, viz., in class (a) the unit matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, in class (b) the matrices for which $\alpha\delta - \beta\gamma = 0$ and $\alpha + \delta = 1$, ∞^2 in number, all similar, whose canonical form is $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, and in class (d) the matrix zero.

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* See §2.

35. *Ternary Matrices.* Denoting a ternary matrix by the symbol (α_{ij}) $(i, j = 1, 2, 3)$, we shall define α'_{ij} as the cofactor of α_{ij} in the determinant $|\alpha_{ij}|$. Ternary matrices may be divided into seven classes, as follows:

$$\text{(a)}\ v = 0,\ r = 3,\ \mu = 1;\ |\alpha_{ij}| \neq 0;$$

∞^9 matrices, non-singular, forming a single entire group.

$$\text{(b)}\ v = 1,\ r = 2,\ \mu = 1;\ |\alpha_{ij}| = 0,\ \sum_1^3 \alpha'_{ii} \neq 0;$$

∞^8 matrices; canonical form $\begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha\delta - \beta\gamma \neq 0$; they are collected into ∞^4 similar entire groups, each containing ∞^4 matrices.

$$\text{(c)}\ v = 2,\ r = 2,\ \mu = 2;\ |\alpha_{ij}| = 0,\ \sum_1^3 \alpha'_{ii} = 0,\ \sum_1^3 \alpha_{ii} \neq 0,$$

at least one first minor $\alpha'_{ij} \neq 0$; ∞^7 matrices; characteristic $[1\,\mathring{2}]$; canonical form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \neq 0$; every matrix has ∞^6 similar matrices, of which ∞^2 are associated with it.

$$\text{(d)}\ v = 2,\ r = 1,\ \mu = 1;\ \alpha'_{ij} = 0\ (i, j = 1, 2, 3),$$

nine equations of condition, of which four are independent, $\sum_1^3 \alpha_{ii} \neq 0$; ∞^5 matrices; characteristic $[1\,(\mathring{1}\,\mathring{1})]$; canonical form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \neq 0$; they form ∞^4 similar entire groups, each containing ∞^1 matrices.

Classes (c) and (d) together consist of ∞^4 similar pseudogroups, the canonical form of which is the set of matrices of the form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$, $\alpha \neq 0$, $ad - bc = a + d = 0$, so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nilpotent or zero. Each of these pseudogroups therefore contains ∞^3 matrices, of which ∞^1 are group-members of class (d) forming an entire group, and the rest are associated non-group-members of class (c); every group-member has ∞^2 associated non-group-members.

$$\text{(e)}\ v = 3,\ r = 2,\ \mu = 3;\ |\alpha_{ij}| = \sum_1^3 \alpha'_{ii} = \sum_1^3 \alpha_{ii} = 0,$$

at least one first minor $\alpha'_{ij} \neq 0$; ∞^6 matrices; characteristic $[\overset{\circ}{3}]$; canonical form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; they are nilpotent, all similar, and their cube is zero.

$$\text{(f)}\ v = 3,\ r = 1,\ \mu = 2;\ \alpha'_{ij} = 0\ (i, j = 1, 2, 3),\ \sum_1^3 \alpha_{ii} = 0,$$

at least one element $\alpha_{ij} \neq 0$; ∞^4 matrices; characteristic $[(\overset{\circ}{2}\ \overset{\circ}{1})]$; canonical form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; they are nilpotent, all similar, and their square is zero.

$$\text{(g)}\ v = 3,\ r = 0,\ \mu = 1;\ \alpha_{ij} = 0\ (i, j = 1, 2, 3);$$

characteristic $[(\overset{\circ}{1}\ \overset{\circ}{1}\ \overset{\circ}{1})]$; one matrix, zero, forming an entire group of order one.

Classes (e), (f), and (g) together form a single pseudogroup, of which zero is the group-member, and all the other matrices are associated nilpotent non-group-members. The square of every matrix of (e) belongs to (f).

There are four kinds of ternary idempotent matrices, viz., in class (a) the unit matrix, in class (b) ∞^4 similar matrices whose canonical form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, in class (d) ∞^4 similar matrices whose canonical form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and in class (g) the matrix zero.

Application to Quaternions.

36. On account of the close connection that exists between the theory of matrices and the theory of hypercomplex numbers, it is clear that the concept of group-membership under multiplication can be transferred from the former to the latter. In particular, it is well-known that quaternions* are abstractly identical with binary matrices, both as to addition and multiplication.† The correspondence between them can be set up by making the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ correspond to the quaternion

$$\frac{\alpha+\delta}{2} - \frac{(\alpha-\delta)\sqrt{-1}}{2}\, i + \frac{\beta-\gamma}{2}\, j - \frac{(\beta+\gamma)\sqrt{-1}}{2}\, k$$

* That is, quaternions of the general kind (sometimes called biquaternions), in which the coefficients of 1, i, j, k are ordinary complex, as well as real, numbers.

† This identification is due to Charles and Benjamin Peirce; see Taber, "On the Theory of Matrices," AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 353.

for all values of α, β, γ, and δ, or, what is the same thing, by making the quaternion $a + bi + cj + dk$ correspond to the matrix

$$\begin{pmatrix} a + b\sqrt{-1} & c + d\sqrt{-1} \\ -c + d\sqrt{-1} & a - b\sqrt{-1} \end{pmatrix},$$

for all values of a, b, c, and d.

By means of this correspondence the classification of binary matrices in § 34 gives rise to a classification of quaternions. Thus every quaternion $a + bi + cj + dk$ belongs to one of four classes, as follows:

$$\text{(a)}\ a^2 + b^2 + c^2 + d^2 \neq 0;$$

∞^4 quaternions, forming a single group simply isomorphic with the entire group of non-singular binary matrices. Its identical element is the idempotent quaternion one.

$$\text{(b)}\ a^2 + b^2 + c^2 + d^2 = 0,\ a \neq 0;$$

∞^3 quaternions, whose canonical form is $\frac{\alpha}{2} - \frac{\alpha\sqrt{-1}}{2} i$ $(\alpha \neq 0)$; they form a doubly infinite system of similar one-parameter groups. The identical elements of these groups are the idempotent quaternions defined by the equations $a^2 + b^2 + c^2 + d^2 = 0$, $a = \frac{1}{2}$; their canonical form is $\frac{1}{2} - \frac{\sqrt{-1}}{2} i$.

$$\text{(c)}\ a = 0,\ b^2 + c^2 + d^2 = 0,$$

at least one of the coefficients $b, c, d \neq 0$; ∞^2 quaternions, all similar; canonical form $\frac{1}{2} j - \frac{\sqrt{-1}}{2} k$; they are nilpotent, and their square is zero; together with zero they form a pseudogroup.

$$\text{(d)}\ a = b = c = d = 0;$$

the quaternion zero, forming a group of order one.

Therefore every quaternion either belongs to a multiplicative group or is nilpotent.

In a similar manner the results of this paper can be applied to other hypercomplex number-systems.

CORNELL UNIVERSITY, *March*, 1908.

Methods to Determine the Primitive Roots of a Number.

By G. A. Miller.

The present note aims to exhibit some elementary relations between well-known methods of finding the primitive roots of a number and the properties of the cyclic group. Incidentally we arrive at a fundamental theorem relating to the primitive roots of a special class of numbers. A corollary of this theorem gives the primitive roots of all the prime numbers of the form $2p+1$, p being a prime, while it has been customary in the works on the theory of numbers to devote two theorems to the primitive roots of such prime numbers.* The note has close contact with the paper published in this Journal under the title "Some Relations between Number Theory and Group Theory" and may be regarded as a continuation of this article.†

It is known that the necessary and sufficient condition that a number g has primitive roots is that the cyclic group G of order g has a cyclic group of isomorphisms I. The numbers which are less than g and prime to it may be made to correspond to the operators of I, unity corresponding to the identity, in such a way that I and the group formed by these numbers, when they are combined by multiplication and the products reduced with respect to modulus g, are simply isomorphic. The orders of the operators of I are the indices of the exponents to which the corresponding numbers belong. In particular, $g-1$ corresponds to the operator of order 2 and the primitive roots of g correspond to the operators of highest order in I. Hence the method of finding the primitive roots of a number is equivalent to that of finding the operators of highest order in a cyclic group.

One of the most instructive methods for finding all the primitive roots of g is analogous to the method known as the "Sieve of Eratosthenes" for finding

* Cf. Cahen, "Éléments de la Théorie des Nombres," 1900, p. 336; Tschebyscheff, "Elemente der Zahlentheorie," 1902, p. 307; Pascal, "Repertorium der höheren Mathematik," Vol. I (1900), p. 580.

† American Journal of Mathematics, Vol. XXVII (1905), p. 315.

the prime numbers which are less than a given number. While there is no general non-tentative method for finding the primitive roots of g, there are general non-tentative methods for finding all the numbers which are less than g and non-primitive roots of g. In group-theory language one of these may be stated as follows: Let p be any prime divisor of g and raise each operator of G to the pth power. These powers are composed of all the operators of G whose orders are not divisible by the highest power of p which divides g. Hence we may obtain all the operators of G which are not of highest order by raising successively all its operators to the powers whose indices are the different prime divisors of g. The remaining operators are of highest order and hence correspond to the primitive roots in I. This method has the advantage that it gives all the primitive roots of g at the same time. Its chief defect is that the same non-primitive root is generally found more than once. From the known congruence $k^2 \equiv (g-k)^2 \bmod g$, it follows that we need to square only half the numbers less than g and prime to g. In particular, when g is a prime number of the form 2^a+1 its non-primitive roots are given by l^2, $l=1, 2, \ldots, 2^{a-1}$. The remaining numbers less than 2^a are its primitive roots.

The most practical general tentative method for finding the primitive roots is based, in group-theory language, upon the fact that the order of the product of two commutative operators is divisible by the highest power of any prime p which divides either one of their orders, provided these orders do not involve the same highest power of p. If we have two commutative operators, we can therefore readily find an operator whose order is the least common multiple of their orders; for, if they should involve the same highest power of p, one of them may be raised to the pth power and thus we can obtain two operators not involving the same highest power of the same prime; and hence the order of their product will be the least common multiple of their orders. When g is not very large, this method generally leads to an operator of highest order in I, or to a primitive root of g, with a few trials. Since all the operators of highest order in any cyclic group may be obtained by raising any one of them to the powers whose indices are prime to this order, all the primitive roots of g may be obtained from any one of them in the same manner.

We are now in position to give a simple proof of the theorem mentioned in the first paragraph. Suppose that the order of I is of the form $2q$, q being any odd prime. Since a square can not correspond to an operator of highest order in I (the order of I being always even) and since α^2, α being any integer prime

to g, can not correspond to the operator of order 2, it follows that α^2 corresponds to an operator of order q whenever $g-1>\alpha>1$. The product of the operator which corresponds to α^2 and the one which corresponds to $g-1$ is of order $2q$, and hence $-\alpha^2$ is a primitive root of g. This result leads directly to the

THEOREM: *When the exponent to which the primitive roots of a given number g belong is of the form $2q$, q being an odd prime, then each of the primitive roots of g is given once and only once by $-\alpha^2$, $1<\alpha<g/2$ and α being prime to g; moreover, α^2 belongs to exponent q.*

COROLLARY: *Every prime of the form $2q+1$, q being an odd prime, has for its primitive roots $-\alpha^2$, $1<\alpha<q+1$. In particular, -4 is a primitive root of every prime of this form.*

It should be observed that the group-theory language employed in the proof of this theorem is not essential, as the results follow directly from the facts that a square can not be a primitive root of any number and that -1 belongs to exponent 2 with respect to any modulus. The forms of the numbers which have primitive roots are assumed to be known throughout the present note. The preceding theorem furnishes the numbers belonging to every possible exponent when g has the required form. It is also clear that the order of I can not be of the form $2q$ unless g is an odd prime, twice an odd prime, 9 or 18.

We may add that some of the known developments in regard to the properties of primitive roots, especially those relating to products, follow directly from the isomorphism between I and the numbers which are less than g and prime to g. For instance, the theorem which affirms that the product of all the primitive roots of a number is congruent to unity whenever the number has more than one primitive root, is included in the evident statement that the continued product of all the operators of the same order >2 in any Abelian group is the identity, since these operators may be arranged in pairs consisting of an operatar and its inverse. The product of all the operators of order 2 in such a group is also known to be the identity whenever the group contains more than one such operator.*

* *Annals of Mathematics*, Vol. IV (1903), p. 188.

*Standard Forms of Certain Types of Peirce Algebras.**

By James Byrnie Shaw.

I. Introduction.

1. The determination of general laws for the relationships of numbers in an algebra of order r has not progressed very far, especially as regards Peirce algebras. By Peirce algebra is meant an algebra containing an idempotent unit η, which is the modulus, all other expressions being nilpotent unless they contain a term of the form $a\eta$, where a is a scalar coefficient. The reduction of algebras in general must depend on such laws of structure of an algebra, and it seems that even in the simpler cases, after we have reduced to forms that have a comparatively simple multiplication table, the complete exhaustion of all the information which can possibly be obtained by using the law of associativity leaves nevertheless a number of arbitrary parameters which can only be removed by linear transformation of the units, if removable at all. The further determination of individual types becomes then somewhat a matter of personal choice. The present paper does not consider this question, which has been touched upon elsewhere. It seeks only to reduce certain particular cases to their simplest forms, thus extending the present narrow list considerably.

2. It is known that, for any Peirce algebra, we may take any one of the nilpotent expressions to be a unit, called the *adjunct unit*,† represented by e_{11}, determine then a set of expressions called the *base*, defined by units η, e_{20}, e_{30}, $\cdots$, e_{m0}. Any expression of the algebra is then linearly expressible in terms of

$$\eta,\ e_{40},\ e_{11},\ e_{40}\,e_{11},\ e_{11}^{2},\ e_{40}\,e_{11}^{2},\ \cdots,\ e_{40}\,e_{11}^{\mu_1-1},\ \cdots,\ e_{11}^{\mu_1-1}.$$

It is preferable of course to choose for e_{11} a number which will give μ_1 as high a

* Read before the Chicago Section of the American Mathematical Society, Dec. 30, 1907.

† Shaw: On Nilpotent Algebras, *Trans. Amer. Math. Soc.* (1903), 4, 405–422.

value as will any other expression in the algebra. In that event, $r - \mu_1$ is called the *deficiency* of the algebra.

3. SCHEFFERS* and others have shown that the units may be *regularised;* that is, put into an order $e_1, \ldots, e_i, \ldots, e_j, \ldots$ such that $e_i e_j$ and $e_j e_i$ are of the form $\Sigma \gamma_{ijk} e_k$ where $k > i$, and $k > j$. The theorem quoted in § 2 implies this.

4. PEIRCE† showed that when the *deficiency is zero,* the algebra is expressible in terms of the units $\eta, e_{11}, \ldots, e_{11}^{r-1}$, or, in a more convenient notation, $\eta, i, i^2, \ldots, i^{r-1}$. He showed further that when the *deficiency is unity,* the algebra is defined by the units $\eta, i, j, \ldots, j^{r-2}$, where

$$ij = 0, \quad i^2 = aj^{r-2}, \quad ji = bj^{r-2}.$$

We find here four types not reducible into each other, namely

I. $a = b = 0.$
II. $a = 0,\ b = 1.$
III. $a = 1,\ b = 0.$
IV. $a = b = 1.$

5. The cases of *deficiency two* were reduced in full by STARKWEATHER.‡ There are three types, each with numerous sub-types. They are as follows.

When $r > 6$.

(1) *Type* $(\eta, i, i^2, j, j^2, \ldots, j^{r-3})$.
(11) $i^3 = j^{r-3}$; $ij = 0 = ji$.
(12) $i^3 = 0$; $ij = 0 = ji$.
(13) $i^3 = j^{r-3}$; $ij = 0$; $ji = 2j^{r-3}$; $ji^2 = 0$.
(14) $i^3 = 0$; $ij = 0$; $ji = 2j^{r-3}$; $ji^2 = 0$.

(2) *Type* $(\eta, i, j, ij, j^2, \ldots, j^{r-3})$.
(21) $i^2 = j^{r-4}$; $i^3 = 0$; $ji = ij$; $i^2j = iji = ji^2 = j^{r-3}$.
(22) $i^2 = j^{r-4}$; $ji = ij + 2j^{r-3}$; $i^2j = j^{r-3} = iji = ji^2$.
(23) $i^2 = j^{r-4}$; $ji = ij + 2j^{r-4} + 2cj^{r-3}$; $i^2j = j^{r-3} = iji = ji^2$; $jij = 2j^{r-3}$.

* SCHEFFERS: *Mathematische Annalen* (1891), 39, 293–390.
† B. PEIRCE: AMER. JOUR. OF MATH. (1881), 4, 97–192.
‡ STARKWEATHER: AMER. JOUR. OF MATH. (1899), 21, 369–386; (1901), 23, 378–402.

In this sub-type $c = 0$ when $r \neq 8$.

(24) $i^2 = j^{r-4} + j^{r-3}$; $ji = -ij - 2j^{r-4}$; $i^2j = j^{r-3}$; $iji = -j^{r-3}$; $ji^2 = j^{r-3}$; $jij = -2j^{r-3}$.

(25) $i^2 = j^{r-4}$; $ji = -ij$; $i^2j = j^{r-3}$; $iji = -j^{r-3}$; $ji^2 = j^{r-3}$.

(26) $i^2 = hj^{r-3}$; $ji = \frac{c}{2-c} ij + 2(2-c) j^{r-4}$; $iji = 0 = i^2j = ji^2$;

$jij = 2(2-c) j^{r-3}$; $j^2i = 4j^{r-3}$.

In this sub-type $h = 0$ or 1 when $r \neq 7$.

(27) $i^2 = j^{r-3}$; $ji = -ij$; $i^2j = 0 = ji^2 = iji = j^2i = jij$.

(28) $i^2 = hj^{r-3}$; $ji = ij + 2j^{r-3}$; $i^2j = 0 = jij = iji = ji^2 = j^2i$.

In this sub-type $h = 0$ or 1 when $r \neq 7$.

(29) $i^2 = j^{r-3}$; $ji = dij$; $i^2j = 0 = iji = ji^2 = j^2i = jij$.

(2α) $i^2 = 0$; $ji = dij$; $i^2j = 0 = iji = ji^2 = j^2i = jij$.

(2β) $i^2 = ij + j^{r-3}$; $ji = 0$; $i^2j = iji = ji^2 = 0 = j^2i = jij = i^3$.

(2γ) $i^2 = ij$; $ji = 0$; $i^2j = iji = ji^2 = 0 = j^2i = jij = i^3$.

(2ε) $i^2 = j^{r-3}$; $ji = 2j^{r-3}$.

(2ζ) $i^2 = j^{r-3}$; $ji = 0$.

(2η) $i^2 = 0 = ji$.

(3) *Type* $(\eta, i, j, k, \ldots, k^{r-3})$.

(31) $i^2 = k^{r-3}$; $ij = ji = 0 = ki = kj = j^2 = jk$.

(32) $i^2 = 0 = ji = ij = ki = ik = jk = kj = j^2$.

(33) $i^2 = 0 = j^2 = ik = ki = jk = kj$; $ij = gk^{r-3}$; $ji = k^{r-3}$.

(34) $i^2 = 0 = j^2$; $ij = k^{r-3} = ji$; $ki = 2k^{r-3} = kj$.

(35) $i^2 = 0 = j^2$; $ij = k^{r-3} = ji$; $kj = 2k^{r-3}$; $ki = 0$.

(36) $i^2 = k^{r-3}$; $j^2 = 0$; $ij = -k^{r-3}$; $ji = k^{r-3}$; $ki = 0 = kj$.

(37) $i^2 = k^{r-3}$; $j^2 = 0$; $ij = 0 = ji$; $ki = 0$; $kj = 2k^{r-3}$.

(38) $i^2 = k^{r-3}$; $j^2 = 0 = ij = ji$; $ki = 2k^{r-3}$; $kj = 0$.

(39) $i^2 = 0 = j^2 = ij = ji = ki$; $kj = 2k^{r-3}$.

The forms which these algebras take when $r = 4$, 5, or 6 appear in SCHEFFERS' * and STARKWEATHER's † lists, and may be found also in SHAW's ‡ "Synopsis."

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* SCHEFFERS: *Math. Ann.* (1891), 39, 293–390.

† STARKWEATHER: AMER. JOUR. OF MATH. (1901), 23, 378–402.

‡ SHAW: "Synopsis of Linear Associative Algebra," Carnegie Institution of Washington, D. C., pp. 103, 105, 106.

6. The theorem mentioned in § 2 may be expressed more definitely by representing the generators in terms of certain ideal units denoted by λ_{ijk}, thus:

$$e_{f0} = \Sigma b_{gh}\lambda_{gh0} + \Sigma a_{ijk}\lambda_{ijk}, \qquad (i, j = 1, 2, \ldots, m)$$
$$e_{11} = \lambda_{111} + \Sigma a'_{ijk}\lambda_{ijk}, \qquad (i, j = 1, 2, \ldots, m)$$
$$\eta = \lambda_{110} + \lambda_{220} + \ldots + \lambda_{mm0},$$

where $\mu_i > k > 1$, and $k \geqq \mu_i - \mu_j$, $g > h$. Further, the coefficients b_{gh} are so chosen that if the terms λ_{ijk} be cut off from the expression for e_{f0}, giving

$$e'_{f0} = \Sigma b_{gh}\lambda_{gh0},$$

then these units define an associative algebra. The ideal units λ_{ijk} satisfy the laws

$$\lambda_{ijk}\lambda_{i'j'k'} = \vartheta_{ji'}\, c\lambda_{ij'\,k+k'},$$

where $c = 1$ when $\mu_i > k + k' \geqq \mu_i - \mu_{j'}$, $k + k' \geqq 0$; otherwise $c = 0$. Also $\vartheta_{ji'} = 1$ if $j = i'$, otherwise $\vartheta_{ji'} = 0$.

In this notation, and starting from this theorem, we may produce the complete set of sub-types of an algebra whose type is assigned. It is purposed to study a few types by this method, both for the results obtained and to show the utility of the method.

II. THE TYPE $(\eta, i, \ldots, i^m, j, \ldots, j^{r-m-1})$.

1. It is implied that $ij = 0$. Since $j^{r-m} = 0$, $\mu_1 = r - m$; since $ij = i^2j = \ldots = i^mj = 0$, $\mu_2 = \mu_3 = \ldots = \mu_{m+1} = 1$. Hence we will find no other forms of λ than $\lambda_{210}, \lambda_{320}, \lambda_{430}, \ldots, \lambda_{m+1,m,0}, \lambda_{111}, \ldots, \lambda_{1,1,r-m-1}, \lambda_{1,t,r-m-1}$ $(t = 2, \ldots, m+1)$.

2. If for convenience we write only the subscripts, omitting λ, then

$$i = (210) + (320) + \ldots + (m+1, m, 0) + \sum_{t=2}^{m+1} a_t(1, t, r-m-1),$$
$$j = (111) + \sum_{t=2}^{m+1} b_t\,(1, t, r-m-1).$$

Thus

$$i^2 = (310) + \ldots + (m+1, m-1, 0) + \sum_{t=2}^{m+1} a_t\,(1, t-1, r-m-1).$$

For $t = 2$, however, $(1, t-1, r-m-1)$ becomes $(1, 1, r-m-1)$ and i^2 would contain a term $a_2 j^{r-m-1}$, which is not possible if $m > 1$. The type for $m = 1$ is included above in I, and need not be discussed again. Hence $a_2 = 0$. Likewise, from $i^3, \ldots, i^m$ we have $a_t = 0$ for $t = 2, 3, \ldots, m$.

Therefore

$$
\begin{aligned}
i \; &= (210) + (320) + \dots + (m+1,\, m,\, 0) + a(1,\, m+1,\, r-m-1); \\
i^2 \; &= (310) + (420) + \dots + (m+1,\, m-1,\, 0) + a(1,\, m,\, r-m-1); \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots; \\
i^m \; &= (m+1,\, 1,\, 0) + a(1,\, 2,\, r-m-1); \\
i^{m+1} &= a(1,\, 1,\, r-m-1).
\end{aligned}
$$

3. Again,

$$
\begin{aligned}
j^2 &= (112), \dots, j^{r-m-1} = (1,\, 1,\, r-m-1); \\
ji &= \sum_{t=2}^{m+1} b_t\,(1,\, t-1,\, r-m-1).
\end{aligned}
$$

But this gives $ji = b_2 j^{r-m-1}$, and $b_t = 0$ for $t > 2$. We have therefore

$$
j = (111) + b(1,\, 2,\, r-m-1).
$$

4. Hence we may suppose finally for all algebras of this type that

$$
\begin{gathered}
i^{m+1} = a j^{r-m-1}; \\
ij = 0; \quad ji = b j^{r-m-1}.
\end{gathered}
$$

If $a \neq 0$, $b = 0$, we may choose $i_1 = i a^{-\frac{1}{m+1}}$, whence

$$
i_1^{m+1} = j^{r-m-1}, \quad j i_1 = 0.
$$

If $a = 0$, $b \neq 0$, we may take $j_1 = j b^{\frac{1}{r-m-2}}$, whence

$$
i^{m+1} = 0, \quad i j_1 = 0, \quad j_1 i = j_1^{\,r-m-1}.
$$

If $a \neq 0 \neq b$, we may take $i_1 = i\left(a^{-1} b^{\frac{r-m-1}{r-m-2}}\right)^{\frac{r-m-2}{m(r-m-2)-1}}$,

$$
j_1 = j(a^{-1}\, b^{m+1})^{\frac{1}{m(r-m-2)-1}}; \text{ whence}
$$

$$
i_1^{m+1} = j_1^{\,r-m-1}, \quad i_1 j_1 = 0, \quad j_1 i_1 = j_1^{\,r-m-1}.
$$

5. We have then four sub-types, given by the equations:

$$
\begin{aligned}
&(1) \quad i^{m+1} = j^{r-m-1}, \quad ji = 0. \\
&(2) \quad i^{m+1} = 0, \qquad\quad\; ji = j^{r-m-1}. \\
&(3) \quad i^{m+1} = j^{r-m-1} = ji. \\
&(4) \quad i^{m+1} = 0 = ji.
\end{aligned}
$$

The types worked out by BENJAMIN PEIRCE for *deficiency unity* are thus extended to all the types for the class represented by the symbol

$$
(\eta,\, i,\, i^2, \dots, i^m, j, j^2, \dots, j^{r-m-1}).
$$

III. THE TYPE $(\eta, i, i^2, \ldots, i^m, j, ij, j^2, \ldots, j^{r-m-2})$.

1. We have now $\mu_1 = r - m - 1$, $\mu_2 = 2$, $\mu_3 = \mu_4 = \ldots = \mu_{m+1} = 1$. Thus

$$i = (210) + \ldots + (m+1, m, 0) + \sum_{t=2}^{m+1} a_t (2, t, 1) + b(1, 2, r-m-3) + \sum_{v=3}^{m+1} c_v (1, v, r-m-2);$$

$$j = (111) + \sum_{t=2}^{m+1} d_t (2, t, 1) + f(1, 2, r-m-3) + \sum_{v=3}^{m+1} g_v (1, v, r-m-2).$$

2. Thus

$$i^2 = (310) + \ldots + (m+1, m-1, 0) + \sum_{t=2}^{m+1} a_t (2, t-1, 1) + b(1, 1, r-m-3) + \sum_{v=3}^{m+1} c_v (1, v-1, r-m-2).$$

Now i^2 is independent of ij or j^{r-m-3}, so that $a_2 = 0 = b$ unless $m = 1$. Since the deficiency would thus be 2, this case has been considered. Likewise we find

$$a_t = 0, \qquad t < m+1; \qquad c_v = 0 \text{ if } v < m+1.$$

Therefore

$$i = (210) + \ldots + (m+1, m, 0) + a(2, m+1, 1) + c(1, m+1, r-m-2);$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

$$i^{m+1} = a(211) + c(1, 1, r-m-2);$$

$$ij = (211).$$

3. Again,

$$j^2 = (112) + f(1, 2, r-m-2) + \sum_{t=2}^{m+1} d_t f(1, t, r-m-2);$$

$$j^3 = (113);$$

$$\ldots\ldots\ldots;$$

$$j^{r-m-3} = (1, 1, r-m-2);$$

$$ji = \sum_{t=2}^{m+1} d_t (2, t-1, 1) + f(1, 1, r-m-3) + fa(1, m+1, r-m-2) + \sum_{v=3}^{m+1} g_v (1, v-1, r-m-2) = d_2(211) + f(1, 1, r-m-3).$$

Therefore

$$d_t = 0 \text{ for } t > 2, \quad fa = 0, \quad g_v = 0 \text{ for } v = 3, \ldots, m+1,$$

and

$$j = (111) + d(221) + f(1, 2, r-m-3),$$

with either

$$f = 0 \text{ or } a = 0.$$

There are then two subdivisions:

(1) $i=(210)+\ldots+(m+1,\,m,\,0)+c(1,\,m+1,\,r-m-2),$
$\quad j=(111)+d(221)+f(1,\,2,\,r-m-3).$

(2) $i=(210)+\ldots+(m+1,m,0)+a(2,m+1,1)+c(1,m+1,r-m-2),$
$\quad j=(111)+d(221).$

In (1), if $c\neq 0$ we may take it equal to 1, and if $f\neq 0$ we may take $f=1$. We may proceed likewise with a and c in (2). Thus we have the sub-types of this type ($m>1$):

$$\left.\begin{array}{lll}
(1) & i^{m+1}=0, & ji=dij; \\
(2) & i^{m+1}=j^{r-m-3}, & ji=dij; \\
(3) & i^{m+1}=ij, & ji=dij; \\
(4) & i^{m+1}=ij+j^{r-m-3}, & ji=dij; \\
(5) & i^{m+1}=0, & ji=dij+j^{r-m-3}; \\
(6) & i^{m+1}=j^{r-m-3}, & ji=dij+j^{r-m-3}.
\end{array}\right\}\ d \text{ arbitrary.}$$

IV. THE TYPE $(\eta,\,i,\,j,\,\ldots,\,j^{m-1},\,k,\,\ldots,\,k^{r-m-1})$.

(*a*) *When Neither i^2 nor ji Contains j^{m-1}.*

1. Here

$$\mu_1=r-m,\ \mu_2=\mu_3=\ldots=\mu_{m+1}=1.$$

Hence

$$i=(210)+\ldots+\sum_{t=2}^{m+1}a_t(1,\,t,\,r-m-1);$$
$$j=(310)+\ldots+(m+1,\,m,\,0)+\sum_{u=3}^{m+1}b_u(1,\,u,\,r-m-1);$$
$$k=(111)+\sum_{v=2}^{m+1}c_v(1,\,v,\,r-m-1).$$

Hence

$$i^2=a_2(1,\,1,\,r-m-1);$$
$$ij=a_3(1,\,1,\,r-m-1)+a_4(1,\,3,\,r-m-1)+\ldots+a_{m+1}(1,\,m,\,r-m-1).$$

Therefore

$$a_t=0\quad\text{for}\quad t>3.$$

Again,

$$ji=b_3(1,\,1,\,r-m-1);$$
$$j^2=(410)+(530)+\ldots+(m+1,\,m-1,\,0)+b_3(1,\,1,\,r-m-1)+b_4(1,\,3,\,r-m-1)+\ldots+b_{m+1}(1,\,m,\,r-m-1).$$

Hence
$$b_3 = 0, \text{ and likewise } b_4 = 0 = b_5 = \ldots = b_m.$$

Again,
$$ki = c_2(1, 1, r-m-1);$$
$$kj = c_3(1, 1, r-m-1) + c_4(1, 3, r-m-1) + \ldots;$$
and
$$c_v = 0 \quad \text{for} \quad v > 3.$$

2. Thence
$$i = (210) + a_2(1, 2, r-m-1) + a_3(1, 3, r-m-1);$$
$$j = (310) + \ldots + (m+1, m, 0) + b_2(1, 2, r-m-1) + b_{m+1}(1, m+1, r-m-1);$$
$$k = (111) + c_2(1, 2, r-m-1) + c_3(1, 3, r-m-1).$$

The defining equations of the algebra become
$$i^2 = a_2 k^{r-m-1}; \qquad j^m = b_{m+1} k^{r-m-1}; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

If $a_2 \neq 0$ we may take $i = a^{\frac{1}{2}} i_1$, which amounts to supposing $a_2 = 1$. Likewise, if $b_{m+1} \neq 0$ we may take $b_{m+1} = 1$. In case $a_2 \neq 0$, $b_{m+1} \neq 0$, $a_3 \neq 0$, we may change i, j, k into such multiples that
$$i^2 = k^{r-m-1}, \qquad j^m = k^{r-m-1}, \qquad ij = k^{r-m-1}.$$

(*b*) *When i^2 Contains j^{m-1}, but ji Does not.*

3. In this case
$$i = (210) + (m+1, 2, 0) + \sum_{t=2}^{m+1} a_t(1, t, r-m-1);$$
$$j = (310) + \ldots + (m+1, m, 0) + \sum_{u=2}^{m+1} b_u(1, u, r-m-1);$$
$$k = (111) + \sum_{v=2}^{m+1} c_v(1, v, r-m-1).$$

Hence
$$i^2 = (m+1, 1, 0) + a_2(1, 1, r-m-1);$$
$$ij = a_3(1, 1, r-m-1) + \ldots;$$
and
$$a_t = 0 \quad \text{for} \quad t > 3.$$

Also
$$ji = b_2(1, 1, r-m-1) + b_{m+1}(1, 2, r-m-1);$$
$$j^2 = (410) + \ldots + (m+1, m-1, 0) + b_3(1, 1, r-m-1) + \ldots + b_{m+1}(1, m, r-m-1).$$

Hence
$$b_u = 0 \quad \text{for} \quad u > 2.$$

Again,

$$ki = c_2(1,\,1,\,r-m-1) + c_{m+1}(1,\,2,\,r-m-1);$$
$$kj = c_3(1,\,1,\,r-m-1) + \ldots;$$

and

$$c_v = 0 \quad \text{for} \quad v > 3.$$

4. We have therefore finally

$$i^2 = j^{m-1} + a_2 k^{r-m-1}, \qquad j^m = 0, \qquad ij = a_3 k^{r-m-1},$$
$$ji = b_2 k^{r-m-1}, \qquad ki = c_2 k^{r-m-1}, \qquad kj = c_3 k^{r-m-1}.$$

(c) *When ji Contains j^{m-1} but i^2 Does not.*

5. $$i = (210) + \sum_{t=2}^{m+1} a_t(1,\,t,\,r-m-1);$$

$$j = (310) + \ldots + (m+1,\,2,\,0) + (m+1,\,m,\,0) + \sum_{u=2}^{m+1} b_u(1,\,u,\,r-m-1);$$
$$k = (111) + \sum_{v=2}^{m+1} c_v(1,\,v,\,r-m-1);$$
$$i^2 = a_2(1,\,1,\,r-m-1);$$
$$ij = a_3(1,\,1,\,r-m-1) + \ldots + a_{m+1}(1,\,m,\,r-m-1) + a_{m+1}(1,\,2,\,r-m-1).$$

Hence

$$a_t = 0 \quad \text{for} \quad t > 3.$$
$$j^2 = (410) + \ldots + (m+1,\,m-1,\,0) + b_3(1,\,1,\,r-m-1) + b_4(1,\,3,\,r-m-1) + \ldots + b_{m+1}(1,\,2,\,r-m-1);$$
$$ji = (m+1,\,1,\,0) + b_2(1,\,1,\,r-m-1).$$

Therefore

$$b_u = 0 \quad \text{for} \quad m+1 > u > 2.$$

Also

$$jij = 0; \quad \text{hence} \quad b_{m+1} = 0.$$

6. Hence, finally,

$$i^2 = a_2 k^{r-m-1}; \qquad j^m = 0; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1} + j^{m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

(d) *When Both i^2 and ji Contain j^{m-1}.*

7. The analysis leads to the equations

$$i^2 = j^{m-1} + a_2 k^{r-m-1}; \qquad j^m = 0; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1} + j^{m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

V. THE TYPE $(\eta, i, j, ij, j^2, ij^2, j^3, \ldots, j^{r-4})$.

1. In this type $\mu_1 = r-3$, $\mu_2 = 3$; hence

$$i = (210) + b(221) + c(222) + d_1(1, 2, r-6) + d_2(1, 2, r-5) + d_3(1, 2, r-4);$$
$$j = (111) + g(221) + h(222) + f_1(1, 2, r-6) + f_2(1, 2, r-5) + f_3(1, 2, r-4).$$

Therefore

$$ij = (211) + bg(222) + d_1g(1, 2, r-5) + (d_2g + d_1h)(1, 2, r-4);$$
$$ij^2 = (211) + d_1g(1, 2, r-4);$$
$$j^2 = (112) + g^2(222) + f_1(g+1)(1, 2, r-5) + (f_2 + f_1h + f_2g)(1, 2, r-4);$$
$$ij^3 = (113) + f_1(g^2+g+1)(1, 2, r-4);$$
$$j^4 = (114).$$

(*a*) *When* $r > 9$.

2. $$i^2 = b(211) + c(212) + b^2(222) + d_1(1, 1, r-6) + d_2(1, 1, r-5) + d_3(1, 1, r-4) + bd_1(1, 2, r-5) + (bd_2 + cd_1)(1, 2, r-4)$$
$$= b(211) + b^2g(222) + bd_1g(1, 2, r-5) + b(d_2g + d_1h)(1, 2, r-4) + c(212) + cd_1g(1, 2, r-4) + d_1(1, 1, r-6) + d_2(1, 1, r-5) + d_3(1, 1, r-4).$$

Thus

$$b^2 = b^2g, \quad bd_1 = bd_1g, \quad bd_2 + cd_1 = bd_2g + bd_1h + cd_1g.$$

3. Again,

$$ji = g(211) + h(212) + f_1(1, 1, r-6) + f_2(1, 1, r-5) + f_3(1, 1, r-4) + bg(222) + (d_1 + bf_1)(1, 2, r-5) + (d_2 + cf_1)(1, 2, r-4)$$
$$= g(211) + bg^2(222) + d_1g^2(1, 2, r-5) + (d_2g^2 + 2d_1gh)(1, 2, r-4) + h(212) + f_1(1, 1, r-6) + f_2(1, 1, r-5) + f_3(1, 1, r-4).$$

Thus

$$bg^2 = bg, \quad d_1 + bf_1 = d_1g^2, \quad d_2 + f_1c = d_2g^2 + 2d_1gh.$$

4. Next,

$$j^2i = g^2(212) + f_1(g+1)(1, 1, r-5) + (f_2 + f_1h + f_2g)(1, 1, r-4) + (d_1 + bf_1(g+1))(1, 2, r-4).$$

Therefore

$$d_1(g^3 - 1) = bf_1(g+1).$$

5. Collecting these results we find the following sub-types:

(1) $b=0,\quad c=0,\quad g\neq\pm 1,\quad d_1=0,\quad d_2=0.$
(2) $b=0,\quad c=0,\quad g=\pm 1,\quad d_1=0.$
(3) $b=0,\quad c=0,\quad g=1,\quad h=0.$
(4) $b=0,\quad c\neq 0,\quad g=1,\quad 2d_1h-cf_1=0.$
(5) $b=0,\quad c\neq 0,\quad g\neq 1,\quad d_1=0,\quad f_1=0.$
(6) $b\neq 0,\quad g=1,\quad d_1=0,\quad f_1=0.$
(7) $b\neq 0,\quad g=1,\quad h=0,\quad f_1=0.$

The resulting equations are

(1) $$\begin{aligned} i^2 &= d_3j^{r-4}; \quad ji=gij+hij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\ j^2i &= g^2ij^2+f_1(g+1)j^{r-5}+(f_3+f_1h+f_2g)j^{r-4};\\ j^3i &= f_1(g+1)j^{r-4};\\ jij &= gij^2+f_1j^{r-5}+f_2j^{r-4};\\ jij^2 &= f_1j^{r-4};\\ j^2ij &= f_1(g+1)j^{r-4}. \end{aligned}$$

(2₁) $$\begin{aligned} i^2 &= d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+hij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\ j^2i &= ij^2+2f_1j^{r-5}+(2f_2+f_1h)j^{r-4}; \quad ji^2=d_2j^{r-4}=i^2j;\\ j^3i &= 2f_1j^{r-4};\\ jij &= ij^2+f_1j^{r-5}+f_2j^{r-4};\\ jij^2 &= f_1j^{r-4};\\ j^2ij &= 2f_1j^{r-4}. \end{aligned}$$

(2₂) $$\begin{aligned} i^2 &= d_2j^{r-5}+d_3j^{r-4}; \quad ji=-ij+h\,ij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\ j^2i &= ij^2+f_1hj^{r-4}; \quad j^3i=0;\\ jij &= -ij^2+f_1j^{r-5}+f_2j^{r-4}; \quad jij^2=f_1j^{r-4}; \quad j^2ij=0;\\ ji^2 &= d_2j^{r-4}=i^2j. \end{aligned}$$

(3) $$\begin{aligned} i^2 &= d_1j^{r-6}+d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\ j^2i &= ij^2+2f_1j^{r-5}+2f_2j^{r-4}; \quad j^3i=3f_1j^{r-4}; \quad jij=ij^2\\ &\qquad +f_1j^{r-5}+f_2j^{r-4};\\ jij^2 &= f_1j^{r-4}; \quad j^2ij=2f_1j^{r-4}; \quad j^3i=2f_1j^{r-4};\\ ji^2 &= d_1j^{r-5}+d_2j^{r-4}=i^2j; \quad j^2i^2=d_1j^{r-4}=i^2j^2. \end{aligned}$$

(4) $$\begin{aligned} i^2 &= ij^2+d_1j^{r-6}+d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+hij^2\\ &\qquad +\frac{2d_1h}{c}j^{r-6}+f_2j^{r-5}+f_3j^{r-4}; \end{aligned}$$

$$jij = ij^2 + \frac{2d_1h}{c}j^{r-5} + f_2j^{r-4}; \quad jij^2 = \frac{2d_1h}{c}j^{r-4};$$
$$j^2i = ij^2 + \frac{4d_1h}{c}j^{r-5} + 2f_2j^{r-4}; \quad j^2ij = \frac{4d_1h}{c}j^{r-4}; \quad j^3i = \frac{4d_1h}{c}j^{r-4};$$
$$ji^2 = d_1j^{r-5} + d_2j^{r-4} = i^2j; \quad j^2i^2 = d_1j^{r-4} = i^2j^2.$$

(5) $i^2 = cij^2 + d_2j^{r-5} + d_3j^{r-4}; \quad ji = gij + hij^2 + f_2j^{r-5} + f_3j^{r-4};$
$jij = gij^2 + f_2j^{r-4}; \quad j^2i = g^2ij^2 + (g+1)f_2j^{r-4};$
$j^2ij = 0; \quad jij^2 = 0; \quad j^3i = 0; \quad ji^2 = d_3j^{r-4} = i^2j.$

(6) $i^2 = bij + cij^2 + d_2j^{r-5} + d_3j^{r-4}; \quad ji = ij + hij^2 + f_2j^{r-5} + f_3j^{r-4};$
$jij = ij^2 + f_2j^{r-4}; \quad jij^2 = 0; \quad i^2j = bij^2 + d_2j^{r-4} = iji;$
$$ji^2 = bij^2 + (bf_2 + d)j^{r-4};$$
$j^2i = ij^2 + 2f_2j^{r-4}; \quad j^2ij = 0 = j^3i; \quad j^2i^2 = 0.$

(7) $i^2 = bij + cij^2 + d_1j^{r-6} + d_2j^{r-5} + d_3j^{r-4}; \quad ji = ij + f_2j^{r-5} + f_3j^{r-4};$
$jij = ij^2 + f_2j^{r-4}; \quad jij^2 = 0 = j^2ij = j^3i;$
$j^2i = ij^2 + 2f_2j^{r-4}; \quad i^2j = bij^2 + d_1j^{r-5} + d_2j^{r-4} = iji;$
$$ji^2 = bij^2 + d_1j^{r-5} + (bf_2 + d_2)j^{r-4};$$
$i^3 = b^2ij^2 + bd_1j^{r-5} + bd_2j^{r-4}; \quad ji^3 = i^3j = bd_1j^{r-4}; \quad i^2j^2 = d_1j^{r-4} = j^2i^2.$

(b) *When* $r = 6$.

6. We have some modifications in the general formulæ. Thus

$$\begin{aligned} i &= (210) + b(221) + c(222) + d(122); \\ j &= (111) + g(221) + h(222) + n(122); \\ ij &= (211) + (n + bg)(222); \\ j^2 &= (112) + g^2(222); \\ ij^2 &= (212). \end{aligned}$$

We find easily
$$i^2 = b(211) + c(212) + (b^2 + d)(222) + d(112).$$
Hence
$$b(n + bg) = -d(g^2 - 1) + b^2.$$
Again
$$ji = g(211) + h(212) + bg(222) + n(112).$$
Hence
$$g(n + bg) + ng^2 = bg, \quad \text{or} \quad (b + n)g^2 = (b - n)g.$$
The sub-types are then

(1) $g = 0, \quad d = bn - b^2.$
$i^2 = bij + cij^2 + (bn - b^2)j^2; \quad ji = hij^2 + nj^2.$
$i^3 = b^2ij^2.$

(2) $g = 1, \quad n = 0.$

$i^2 = bij + cij^2 + dj^3; \quad ji = ij + hij^2.$

$i^3 = b^2 ij^2.$

(3) $g = -1, \quad b = 0.$

$i^2 = cij^2 + dj^3; \quad ji = -ij + hij^2 + nj^3.$

$i^3 = 0.$

(4) $g \neq 0, \quad g \neq \pm 1.$

$i^2 = bij + cij^2 + \frac{bng}{g^2 - 1} j^3; \quad ji = gij + hij^2 + nj^3.$

$i^3 = b^2 ij^2.$

(c) *When* $r = 7$.

7. In this case

$$\begin{aligned} i &= (210) + a(221) + b(222) + c(122) + d(123); \\ j &= (111) + f(221) + g(222) + h(122) + l(123); \\ ij &= (211) + (h + af)(222) + cf(123); \\ j^2 &= (112) + f^2(222) + (f + 1)h(123); \\ ij^2 &= (212); \\ j^3 &= (113). \end{aligned}$$

Then

$$\begin{aligned} i^2 &= a(211) + b(212) + (a^2 + c)(222) + c(112) + d(113) + ac(123) \\ &= a(211) + a(h + af)(222) + acf(123) + c(112) + cf^2(222) \\ &\qquad + c(f + 1)h(123) + b(212) + d(113). \end{aligned}$$

Hence

$$a^2 + c = cf^2 + a^2 f + ah, \qquad (h + a)cf + ch = ac.$$

Again,

$$ji = f(211) + g(212) + af(222) + (ah + c)(123) + h(112) + l(113).$$

Hence

$$\begin{aligned} fh + af^2 + f^2 h &= af, \\ cf^2 + (f + 1)h^2 &= c + ah. \end{aligned}$$

The sub-types are therefore

(1) $f = 0, \quad a = h, \quad c = 0. \qquad i^2 = aij + dj^3; \quad ji = gij^2 + lj^3.$

(2) $f \neq 0, \quad a = 0 = c = h. \qquad i^2 = dj^3; \quad ji = fij + gij^2 + lj^3.$

(3) $f = \frac{a - h}{a + h}, \quad a \neq h, \quad c = \frac{1}{4}(h^2 - a^2).$

$i^2 = aij + \frac{1}{4}(h^2 - a^2)j^2 + dj^3.$

$ji = \frac{a - h}{a + h} ij + gij^2 + lj^3.$

(*d*) *When* $r = 8$.

8. We find, in the same way as before,

$$
\begin{aligned}
i &= (210) + a(221) + b(222) + c(122) + d(123) + e(124);\\
j &= (111) + f(221) + g(222) + h(122) + k(123) + l(124);\\
ij &= (211) + (af + h)(222) + cf(123) + (cg + df)(124);\\
j^2 &= (112) + f^2(222) + (1 + f)h(123) + (1 + f)k(124);\\
ij^2 &= (212) + cf^2(124);\\
j^3 &= (113) + h(1 + f + f^2)(124);\\
j^2 &= a(211) + b(212) + c(112) + (a^2 + c)(222) + d(113)\\
&\qquad + e(114) + ac(123) + (ad + bc)(124)\\
&= a(211) + (a^2 f + ah)(222) + acf(123) + (acg + adf)(124)\\
&\qquad + b(212) + bcf^2(124) + c(112) + cf^2(222) + (ch + cfh)(123)\\
&\qquad + (ck + cfk)(124) + d(113) + dh(f^2 + f + 1)(124) + e(114).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&a^2(f - 1) + c(f^2 - 1) + ah = 0,\\
&ac(f - 1) + ch(f + 1) = 0,\\
&ad(f - 1) + bc(f^2 - 1) + ck(f + 1) + acg + dh(f^2 + f + 1) = 0.
\end{aligned}
$$

Again

$$
\begin{aligned}
ji &= f(211) + g(212) + h(112) + k(113) + l(114) + af(222)\\
&\qquad + (c + ah)(123) + (d + ak + bh)(124)\\
&= f(211) + (af^2 + fh)(222) + cf^2(123) + (cfg + df^2)(124)\\
&\qquad + g(212) + cf^2 g(124) + h(112) + f^2 h(222) + (h^2 + fh^2)(123)\\
&\qquad + (hk + fhk)(124) + k(113) + kh(f^2 + f + 1)(124) + l(114).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&af(f - 1) + hf(f + 1) = 0,\\
&c(f^2 - 1) + h^2(f + 1) - ah = 0,\\
&cgf(f + 1) + d(f^2 - 1) + kh(f^2 + 2f + 2) - ak - bh = 0.
\end{aligned}
$$

From i^3 we find

$$(a^2 + c)c(f^2 - 1) + ach(f^2 + f + 1) = 0.$$

From $j^2 i$ we find

$$c(f^4 - 1) - ah(f + 1) + h^2(f + 1)(f^2 + f + 1) = 0.$$

This yields twelve sub-types:

(1) $c = 0 = a = d, \quad h \neq 0, \quad f = -1, \quad b = k.$

$$
\begin{aligned}
i^2 &= bij^2 + ej^4;\\
ji &= -ij + gij^2 + hj^2 + bj^3 + lj^4.
\end{aligned}
$$

(2) $c = 0 = h = k, \quad a \neq 0, \quad f = 1.$
$i^2 = aij + bij^2 + dj^3 + ej^4;$
$ji = ij + gij^2 + lj^4.$

(3) $c = 0 = f, \quad h = a \neq 0, \quad d = a(k - b).$
$i^2 = aij + bij^2 + a(k-b)j^3 + ej^4;$
$ji = gij^2 + aj^2 + kj^3 + lj^4.$

(4) $c = 0 = a = h = d.$
$i^2 = bij^2 + ej^4;$
$ji = fij + gij^2 + kj^3 + lj^4.$

(5, 6) $c = 0 = a = h, \quad f = \pm 1, \quad d \neq 0.$
$i^2 = bij^2 + dj^3 + ej^4.$
$ji = \pm ij + gij^2 + kj^3 + lj^4.$

(7) $a = 0, \quad h \neq 0, \quad f = -1, \quad d = 0, \quad b = k.$
$i^2 = bij^2 + cj^2 + ej^4;$
$ji = -ij + gij^2 + hj^2 + bj^3 + lj^4.$

(8) $a \neq 0, \quad h = 0, \quad f = 1, \quad g = 0, \quad a \neq 2\sqrt{-c}.$
$i^2 = aij + bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + kj^3 + lj^4.$

(9) $a \neq 0, \quad h = 0, \quad f = 1, \quad g \neq 0, \quad a = 2\sqrt{-c}.$
$i^2 = 2\sqrt{-c}\, ij + bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + gij^2 + kj^3 + lj^4.$

(10) $a = 0 = h, \quad f = -1.$
$i^2 = bij^2 + cj^2 + dj^3 + ej^4;$
$ji = -ij + gij^2 + kj^3 + lj^4.$

(11) $a = 0 = h = g = k, \quad f = 1.$
$i^2 = bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + lj^4.$

(12) $h = \pm\sqrt{-1}\, a \neq 0, \quad f = \mp\sqrt{-1}, \quad c = -\tfrac{1}{2}a^2,$
$b = \tfrac{1}{2}ag(1 - \sqrt{-1}), \quad d = \tfrac{1}{2}a^2g + \tfrac{1}{2}ak(1 - \sqrt{-1}).$
$i^2 = aij + \tfrac{1}{2}ag(1 - \sqrt{-1})\, ij^2 - \tfrac{1}{2}a^2j^2$
$\qquad + (\tfrac{1}{2}a^2g + \tfrac{1}{2}ak(1 - \sqrt{-1}))\, j^3 + ej^4;$
$ji = \mp\sqrt{-1}\, ij + gij^2 \pm \sqrt{-1}\, aj^2 + kj^3 + lj^4.$

(e) *When* $r = 9$.

9. $i = (210) + a(221) + b(222) + c(123) + d(124) + e(125)$;
$j = (111) + f(221) + g(222) + h(123) + k(124) + l(125)$;
$ij = (211) + af(222) + cf(124) + (cg + df)(125)$;
$j^2 = (112) + f^2(222) + h(1+f)(124) + (k + kf + hg)(125)$;
$ij^2 = (212) + cf^2(125)$;
$j^3 = (113) + h(f^2 + f + 1)(125)$.

Then

$$i^2 = a(211) + a^2(222) + ac(124) + (bc + ad)(125) + b(212) + c(113) + d(114).$$

Hence

$$a^2(f-1) = 0, \quad ac(f-1) = 0,$$
$$agc + ad(f-1) + bc(f^2 - 1) + ch(f^2 + f + 1) = 0.$$

Again

$$ji = f(211) + af(222) + c(124) + d(125) + h(113) + ah(124) + bh(125) + k(114) + ak(125).$$

Hence

$$af(f-1) = 0, \quad af^2c - c - ah = 0,$$
$$cfg + f^2d + h^2(f^2 + f + 1) = d + bh + ak.$$

Again

$$i^3 = a^2(212) + a^2c(125) + ac(114) + (bc + ad)(115).$$

Hence

$$a^2c(f^2 - 1) = 0.$$

Again

$$j^2i = f^2(212) + (c + ah + afh)(125) + h(f+1)(114) + k(f+1)(115).$$

Hence

$$cf^4 = afh + ah + c.$$

These equations reduce to

$$a(f-1) = 0,$$
$$agc + bc(f^2 - 1) + ch(f^2 + f + 1) = 0,$$
$$af^2c - c - ah = 0,$$
$$c(f^4 - 1) = 0,$$
$$fgc + f^2d + h^2(f^2 + f + 1) - d - bh - ak = 0.$$

The sub-types are

(1) $a = 0 = c, \quad f = 1, \quad h = 0.$
$i^2 = bij^2 + dj^4 + ej^5$;
$ji = ij + kj^4 + lj^5$.

(2) $a = 0 = c, \quad f = 1, \quad h \neq 0, \quad b = 3h.$
$i^2 = 3hij^3 + dj^4 + ej^5;$
$ji = ij + hj^3 + kj^4 + lj^5.$

(3) $a \neq 0, \quad f = 1, \quad c = 0 = h = k.$
$i^2 = aij + bij^2 + dj^4 + ej^5;$
$ji = ij + lj^5.$

(4) $a \neq 0 \neq g, \quad f = 1, \quad c = -\dfrac{a^2 g}{3(a-1)},$

$h = -\tfrac{1}{3}ag, \quad k = -\dfrac{ag^2}{3(a-1)} + g(3ag + b).$

$i^2 = aij + bij^2 - \dfrac{a^2 g}{3(a-1)} j^3 + dj^4 + ej^5;$

$ji = ij - \tfrac{1}{3}agj^3 + \left(\dfrac{-ag^2}{3(a-1)} + g(3ag + b)\right) j^4 + lj^5.$

*On the Law of Gravitation in the Binary Systems.**

By Frank Loxley Griffin.

§ 1. *The Problem.*

That the Newtonian law of attraction operates in the solar system, was long ago shown to be very highly probable; the discovery of binary systems naturally raised the question as to whether the same law operates in them. Any reply to such a question is based, of course, upon both the observational data and certain fundamental assumptions expressed or implied. In this paper the assumptions hitherto used in demonstrating the operation of Newton's law are replaced by others; incidentally a new proof with the old hypotheses is obtained.

The observational data have to do with the apparent orbit, which is merely the projection of the actual orbit upon the plane tangent to the "celestial sphere"; but, combined with the assumption that the actual orbit is a plane curve, they permit the conclusion that the satellite describes an ellipse, subject to the action of a central force directed toward the primary.†

These data being accepted, the operation of Newton's law can be established by applying a theorem of Darboux and Halphen ‡ and employing the following assumptions: the force

(I) is a function of the distance alone,

(II) does not vanish at the center of force, and

(III) admits no orbits except conic sections, whatever be the initial conditions.

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* Read (in part) before the American Mathematical Society December 28, 1906, and (supplemented) September 6, 1907.

† F. Tisserand: *Traité de Mécanique Céleste*, Vol. I, p. 35. In the last analysis, this conclusion is to be regarded as a pure assumption. For, in the first place, observational error is considerable in comparison with the small angles measured; so that the observed positions often deviate widely from even the "most probable ellipse". And secondly, — and more important, — even if absolutely correct observations were possible, any number of them would, without the help of assumptions, be powerless to give any information as to the positions of the star at any instants save those of the observations, or to show that the orbit is a conic with the law of areas exactly fulfilled.

‡ *Comptes rendus de l'Académie des Sciences* (Institut de France), Vol. LXXXIV, pp. 760–62, 936–38, 939–41; cf. also Tisserand, *loc. cit.*, pp. 36–42.

Another demonstration uses the following assumptions with a theorem of Bertrand:* the force

(I) is a function of the distance alone, and

(I a) is a single-valued function, †

(II) does not vanish at the centre of force, and

(III) admits no orbits except closed curves whatever be the initial conditions, provided that the velocity be less than a certain limit.

For the proof given in the present paper the hypotheses are: the force

(I) is a function of the distance alone,

(II) does not vanish at the centre of force, and

(III) varies along the given ellipse according to a law which, if operating throughout the plane, *would permit real motion in some region* outside of any circle about the center of force.

In the present paper, as in the classic discussions, the bodies are regarded as material points. [See § 15, however.]

In comparing these three sets of hypotheses, it is hardly necessary to point out the great severity of the third in each of the classic sets [that of Darboux and Halphen being more severe than that of Bertrand, involving the latter]; or to notice the mildness of the third in the new set, since to deny it is to admit either that the force varies according to different laws in different parts of the plane, or that a body started anywhere outside of a certain circle with real initial conditions would immediately disappear from space. Also (I a), while of the sort readily admitted in considering the forces of nature, is one of the least easily dispensed with in the new set, its absence necessitating an increased use of the remaining hypotheses and a considerably lengthened discussion. It should, however, be noted that (II) can be omitted from either of the classic sets, if to the observational data be added the statement, probably justifiable, that the primary star is not at the center of the ellipse described by the satellite. And further, — because of a theorem due to Bertrand,‡ — Darboux and Halphen,

* *Comptes rendus*, Vol. LXXVII, pp. 849–53; cf. Tisserand, *loc. cit.*, pp. 43–48.

† This is tacitly used by Bertrand in the assumption that the orbit has only one apsidal angle. For a simple example of an orbit with two apsidal angles see a paper by the writer in the *American Mathematical Monthly*, Vol. XIV (1907), pp. 199–201.

‡ *Comptes rendus*, Vol. LXXXIV, pp. 731–32.

knowing that at least one orbit is an ellipse, do not need that part of the observational data which states that the force is central.

The present discussion has three parts: (1) a proof of the operation of the Newtonian law; (2) a treatment of the trajectories admitted by the laws satisfying hypotheses (I) and (III),—an extensive category; and (3) a consideration of various questions relating to sets of hypotheses, including a new proof with the hypotheses of Darboux and Halphen.

The demonstration in part (1) starts from the fact that the force required for the description of the given ellipse as a central orbit varies in a manner dependent upon the position of the center of force in the plane. From the polar equation of the ellipse, the pole being at any point, an expression is obtainable for the required force as a function of the distance alone. An examination of the equation of the ellipse shows that if the center of force be elsewhere than on the major axis, hypothesis (III) is contradicted; and if the center lies elsewhere than at a focus, (II) is not satisfied.

It may be noted at this point that the center of force must lie *within* the given ellipse, since a closed curve can not be described as a central orbit with a center of force exterior to, or upon, the curve.*

§ 2. *Lemma Concerning Imaginary Forces.*

For later reference it is well to observe that if the resultant force—central or not—be imaginary in any portion of space, real motion is impossible in that region. For, suppose a particle (x, y, z) to move there, that is, let

$$x = \phi_1(t), \quad y = \phi_2(t), \quad z = \phi_3(t),$$

where the $\phi_i(t)$ are *real* functions. From the definition of a derivative it follows that $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are real functions of t, as are also the second derivatives, or component accelerations. Hence, at every instant, the resultant acceleration and force are real. This contradicts the hypothesis, so that the supposition of real motion must be abandoned.

* E. J. Routh: *Dynamics of a Particle*, pp. 292–93.

§ 3. *Polar Equation of the Given Ellipse.*

Let the center of force be any point within the given ellipse but not on a principal axis. Let coördinate axes coinciding with the principal axes be so chosen that the center of force (x_1, y_1) lies in the third quadrant. Then the equation of the ellipse is

$$(1-e^2)x^2 + y^2 = a^2(1-e^2), \tag{1}$$

where a and e are respectively the major semi-axis and the eccentricity. Now from any point (x_1, y_1) of the third quadrant there is a normal to the ellipse (in the first quadrant), where the radius vector is a maximum, and where, if λ denote the angle between the X-axis (positively directed) and the normal (outward directed), $0 < \lambda < \frac{\pi}{2}$. Let ρ_1 be the distance from the foot of the normal (x_0, y_0) to (x_1, y_1), and ρ_2 be the normal length to the Y-axis. Then, by hypothesis, $\rho_1 > \rho_2$.

Let the center of force be selected now as pole, and the coördinates of any point of the ellipse be u and θ, where u is the reciprocal of the radius vector ρ, and θ is the longitude measured from a line parallel to OX. Then, since

$$x = x_1 + \rho \cos\theta, \qquad y = y_1 + \rho \sin\theta,$$

equation (1) becomes

$$(1-e^2)(\cos\theta + x_1 u)^2 + (\sin\theta + y_1 u)^2 = a^2(1-e^2)u^2. \tag{2}$$

The elimination of $\sin\theta$ gives

$$(\alpha\cos^2\theta + \beta\cos\theta + \gamma)^2 = \delta^2(1-\cos^2\theta), \tag{3}$$

where

$$\alpha = -e^2, \quad \beta = 2x_1 u(1-e^2), \quad \gamma = 1 + y_1^2 u^2 + (x_1^2 - a^2)(1-e^2)u^2,$$

and

$$\delta = 2y_1 u.$$

Re-arrangement of (3) yields the desired form:

$$\cos^4\theta + 4B\cos^3\theta + 6C\cos^2\theta + 4D\cos\theta + E = 0, \tag{4}$$

where

$$B = \frac{\beta}{2\alpha}, \quad C = \frac{\beta^2 + 2\alpha\gamma + \delta^2}{6\alpha^2}, \quad D = \frac{\beta\gamma}{2\alpha^2}, \quad E = \frac{\gamma^2 - \delta^2}{\alpha^2}.$$

§ 4. *θ as a Function of u.*

It is necessary to examine the solutions of the quartic equation (4).* Now this equation is satisfied by $\cos\theta = \cos\lambda$ if $u = u_1 = \frac{1}{\rho_1}$. Since u is a minimum at $\theta = \lambda$, there are not only values of θ greater than λ but also values of θ less than λ, for which u has any value slightly greater than u_1. Hence equation (4) defines $\cos\theta$ as a multiple-valued function of u; that is, $\cos\theta$ is one function of u one side of the maximum normal and another function the other side.

But $\cos\theta$ is an *algebraic* function of u; viz.:

$$\cos\theta = -B \pm \sqrt{p} \pm \sqrt{q} \pm \sqrt{r}, \tag{5}$$

in which p, q and r are the roots of Euler's cubic, and the product of the radicals has the sign of $-G$, where $G = D - 3BC + 2B^3 = -\frac{\beta\delta^3}{4\alpha^3}$. Since x_1 and y_1 are negative, so are β and δ and consequently G. Hence either all three radicals in (5) must be positive or one positive and two negative; also, none of the roots p, q, r, can vanish. These latter are given by

$$\begin{aligned} p &= B^2 - C + \frac{1}{2}\sqrt[3]{-J+\sqrt{M}} + \frac{1}{2}\sqrt[3]{-J-\sqrt{M}}, \\ q &= B^2 - C + \frac{\omega}{2}\sqrt[3]{-J+\sqrt{M}} + \frac{\omega^2}{2}\sqrt[3]{-J-\sqrt{M}}, \\ r &= B^2 - C + \frac{\omega^2}{2}\sqrt[3]{-J+\sqrt{M}} + \frac{\omega}{2}\sqrt[3]{-J-\sqrt{M}}, \end{aligned} \tag{6}$$

where in each case the radical denotes the principal root, and where

$$\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \text{ and } -27M = I^3 - 27J^2 \text{ (the discriminant)},$$

I and J being certain integral functions of the coefficients B, C, D and E.

Now since $\cos\theta$ is an algebraic function of u, it can be double-valued only by some radical having different signs on the two sides of the maximum normal. This requires that one of the outer radicals $\sqrt{p}$, $\sqrt{q}$, or $\sqrt{r}$ shall change sign; for in the definitions (6) each sign has been definitely chosen. But, since G is negative always, this requires that two of the radicals shall change sign. Moreover, in the ellipse $\cos\theta$ is a continuous function of the continuous variable u,

* In this section the notation is that of Burnside and Panton in *Theory of Equations*, p. 121.

and as p, q and r can not vanish, the two radicals which change must be always of opposite signs, and their difference must vanish for $u = u_1$.

Now these two radicals are $\sqrt{q}$ and $\sqrt{r}$, and these are distinct from $\sqrt{p}$ for $u = u_1$, which follows from conditions for equal roots of Euler's cubic: $I^3 - 27J^2 = 0$ for two equal roots, and $I = J = 0$ for three equal roots.* It has been seen that the first condition must be satisfied for $u = u_1$; (6) then shows that q and r become equal. That the latter conditions are not satisfied may be seen by a laborious direct computation of I or J, which may be avoided as follows: If $p = q = r$ for $u = u_1$, the quartic equation (4) will have a triple root, $\cos\theta = \cos\lambda$, as three of the four possible combinations of signs in (5) yield the same value. But while $\cos\theta = \cos\lambda$ is a double solution regardless of the position of the origin on the normal, the solution is triple only if the origin lie on the major axis or at the center of curvature for the given normal.† Both of these possibilities are excluded, since $\rho_1 > \rho_2 >$ radius of curvature $>$ normal length to x-axis. Hence, for the positions of the center of force at present admitted, $I \cdot J \neq 0$ for $u = u_1$.

Finally, the proper signs being chosen, (5) becomes

$$\cos\theta = -B - \sqrt{p} \pm (\sqrt{q} - \sqrt{r}), \tag{7}$$

where one sign is required when $0 < \theta < \lambda$, and the other sign when $\lambda < \theta < \frac{\pi}{2}$.

The next step of importance is to show that the difference $\sqrt{q} - \sqrt{r}$ contains the factor $(u - u_1)^{\frac{1}{2}}$; *i. e.*, $\sqrt{q} - \sqrt{r} = \sqrt{u - u_1} \cdot \psi(u)$ where $\underset{u=u_1}{L}\, \psi(u) \neq 0, \infty$. For this purpose let $\sqrt{q} - \sqrt{r}$ be expressed as a fraction, thus:

$$\sqrt{q} - \sqrt{r} = \frac{\sqrt{-3M}}{(\sqrt{q} + \sqrt{r})\,[(-J + \sqrt{M})^{\frac{1}{3}} + \frac{I}{3} + (-J - \sqrt{M})^{\frac{2}{3}}]},$$

which shows, since every term in each factor of the denominator is positive at $u = u_1$, that $\sqrt{q} - \sqrt{r}$ contains the factor $u - u_1$ to the same power as does $\sqrt{M}$,

* Burnside and Panton, *loc. cit.*, pp. 124, 144.

† For the circle $u = u_1$ meets the ellipse in only two coincident points at (u_1, λ), unless the circle be the osculating circle for that normal, in which case they meet in three coincident points. If the pole be upon OX, the double solutions $\cos\theta = \cos\lambda$, and $\cos\theta = \cos(-\lambda)$, make either a quadruple solution.

Verification may be obtained by the test with the derived equations after expressing x_1 and y_1 as functions of λ. (See a problem, by the author, in the *American Mathematical Monthly*, Jan. 1909.)

Since M is a polynomial (of degree 12) in u, vanishing for $u=u_1$, some integral power of $u-u_1$ is a factor. To test directly the order of this factor is unnecessary, as it will appear in the next equation.

Differentiate (7) with respect to u, denoting derivatives by primes; substitute $\sqrt[3]{-J+\sqrt{M}}=R_1$, $\sqrt[3]{-J-\sqrt{M}}=R_2$, $\frac{1}{\sqrt{q}}-\frac{1}{\sqrt{r}}=2U$, and $\frac{1}{\sqrt{q}}+\frac{1}{\sqrt{r}}=\frac{2V}{\sqrt{3}}$, so that $\frac{\omega^2}{\sqrt{r}}-\frac{\omega}{\sqrt{q}}=U-iV$, and $\frac{\omega}{\sqrt{r}}-\frac{\omega^2}{\sqrt{q}}=U+iV$; and re-arrange:

$$\begin{aligned}\sin\theta\frac{d\theta}{du}=B'+\frac{2BB'-C'}{2}\left[\frac{1}{\sqrt{p}}\mp 2U\right]-\frac{J'}{12}\left\{\left(\frac{1}{\sqrt{p}}\pm U\right)\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right.\\ \left.\mp iV\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\right\}+\frac{M'}{24\sqrt{M}}\left\{\left(\frac{1}{\sqrt{p}}\pm U\right)\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\mp iV\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right\}.\end{aligned}\tag{8}$$

Now since the line $\theta=\lambda$ is a normal, $\frac{du}{d\theta}=0$ at (u_1, λ). The only denominator in (8) which vanishes at $u=u_1$ is $\sqrt{M}$. If M contains the factor $(u-u_1)^\nu$, M' contains the factor $(u-u_1)^{\nu-1}$; hence, to have $\frac{du}{d\theta}=0$ for $u=u_1$, it is necessary that $\frac{\nu}{2}>\nu-1$, or $\nu<2$. Therefore $\nu=1$. Also, putting $\sqrt{u-u_1}=x$, inspection shows that x occurs to the power unity in U and in $\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)$, and to the power zero in V, B', $2BB'-C'$, and $\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)$; $(2BB'-C')$ may however vanish identically). Whether J' carries a factor x is immaterial; suppose x appears to the power $(n-2)$.

Then equation (8) may be written in the form

$$\sin\theta\,\frac{d\theta}{du}=\frac{1}{x}\left(g+hx+kx^2+lx^n\right),\tag{9}$$

where g, h, k and l are functions of u, which do not vanish at $u=u_1$ unless identically zero. Moreover, from (7) one obtains

$$\begin{aligned}\sin^2\theta&=1-\left[(B+\sqrt{p})^2\mp 2(B+\sqrt{p})(\sqrt{q}-\sqrt{r})+(\sqrt{q}-\sqrt{r})^2\right]\\&=b+cx-x^2,\end{aligned}\tag{10}$$

where b and c are functions of u, not vanishing with $u - u_1$, and not identically zero since $\theta = \lambda \neq 0, \frac{\pi}{2}$, when $u = u_1$. Hence

$$\left(\frac{du}{d\theta}\right)^2 = x^2 \frac{b + cx - x^2}{(g + hx + kx^2 + lx^n)^2}. \tag{11}$$

The numerator of (11) contains an odd power of x. Even if the numerator is a factor of the denominator, the removal of this factor will still leave the same factor in the denominator, as the latter is a perfect square. And the conjugate factor needed for removing odd powers, if present at all, is squared. Therefore, (11) contains essentially an odd power of x.

Finally, then, for $u < u_1$, x being a pure imaginary, $\left(\frac{du}{d\theta}\right)^2$ is a complex function of u; thus

$$\left(\frac{du}{d\theta}\right)^2 = \Phi(u) = F_1(u) + iF_2(u), \tag{12}$$

where F_1 and F_2 are real for $u < u_1$, but $iF_2(u)$ is real for $u > u_1$.

§ 5. *Exclusion of Laws for Non-Axial Centers.*

The foregoing results permit a great restriction upon the location of the center of force within the ellipse. As is well known the force necessary for the description of any curve as a central orbit, the center of force being at the pole, is given at all points of the curve by

$$f = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2}\right),$$

where h is the constant of areas. There are an infinite number of laws according to any of which the force may vary to produce a given orbit with a given constant of areas and a given position of the center of force;* but by imposing hypothesis (I) at the start, a single law is obtained for the variation of the force along

*Cf. F. L. Griffin: Certain Trajectories Common to Different Laws of Central Force, *The Astronomical Journal*, Vol. XXVI (1908), pp. 3–4. Inattention to this fact seems responsible for the use of hypotheses inadequate for the conclusion stated concerning binary systems by Professor Moulton [*Introduction to Celestial Mechanics*, p. 80]. In this connection the writer desires to acknowledge his indebtedness to Professor Moulton for much helpful criticism of this paper, which had its origin in the detection of the error mentioned.

the orbit as a function of u alone. Thus, for the positions of the center of force considered above, the differentiation of (12) gives

$$f = \frac{1}{2} h^2 u^2 [2u + \Phi'(u)].$$

If the force varies according to this law throughout the plane, the resultant force is imaginary when $u < u_1$, that is, everywhere outside of the circle $u = u_1$ [or $\rho = \rho_1$]. By the lemma above, real motion is impossible without that circle, so that hypothesis (III) is contradicted. This law and the corresponding positions of the center are, therefore, excluded; that is, the center of force must lie upon a principal axis.

[It may be observed in passing that the foregoing demonstration applies to exterior points and to those upon the ellipse as well as to interior points, and might have been used to exclude such points, if $0 < \lambda < \frac{\pi}{2}$. Further it excludes points of the minor axis within the evolute (except the center), since from such a point there is a non-axial maximum normal. But in some ellipses the evolute does not contain the entire minor axis.]

§ 6. *Exclusion of Laws for Centers on the Minor Axis.*

Next suppose the center of force to be any point $(0, d)$ of the minor axis Then, choosing that point as the pole and introducing polar coordinates, equation (2) becomes, since $x_1 = 0$, $y_1 = d$:

$$e^2 \rho^2 \sin^2\theta + 2d\rho \sin\theta + d^2 + (1 - e^2)(\rho^2 - a^2) = 0. \tag{13}$$

The solution of (13) involves $\sqrt{(1-e^2)(d^2 + a^2 e^2 - e^2 \rho^2)}$, and so does the expression for the force (unless $d = 0$), as may be seen by forming $\frac{d^2 u}{d\theta^2}$. Now this radical is imaginary for $\rho > \frac{1}{e}\sqrt{d^2 + a^2 e^2}$, so that real motion is impossible outside a certain circle. Again there is a contradiction of hypothesis (III), restricting the center of force to lie upon the major axis. [It may be noted that points in the line of the minor axis exterior to, or upon, the ellipse are likewise excluded.]

§ 7. *Laws Admitted by Hypotheses (I) and (III).*

Finally, let the center of force be any point $(d, 0)$ of the major axis. [There is no loss of generality in restricting d to positive values, as this amounts merely to a choice of the positive direction of an axis.] Equation (2) becomes in this case

$$e^2\rho^2\cos^2\theta - 2d\rho(1-e^2)\cos\theta + (1-e^2)(a^2-d^2) - \rho^2 = 0. \tag{14}$$

The solution of (14) is

$$\cos\theta = \frac{1}{\rho e^2}\left[d(1-e^2) \pm \sqrt{e^2\rho^2 + (d^2-a^2e^2)(1-e^2)}\right]. \tag{15}$$

If $d > ae$, the radical does not vanish for real values of ρ. Its proper sign is ascertained by substituting the coördinates of a particular point, $\theta = 0$, $\rho = a - d$:

$$1 = \frac{d(1-e^2) \pm (d - ae^2)}{e^2(a-d)} \tag{16}$$

which requires that the lower sign be chosen. Equation (15) may then be written

$$\gamma\cos\theta = \alpha u - \sqrt{\beta u^2 + \gamma}, \tag{17}$$

where

$$\alpha = d(1-e^2), \qquad \beta = (d^2 - a^2e^2)(1-e^2), \qquad \gamma = e^2. \tag{18}$$

If $d < ae$, the radical vanishes; but the corresponding value of ρ, viz., $\rho = \sqrt{-\frac{\beta}{\gamma}}$, is not taken in the ellipse unless $d \overline{\gtrless} ae^2$. Tests similar to (16) show that

when $ae^2 \overline{\gtrless} d < ae$, the sign of the radical is always —, and
when $d < ae^2$, the sign is + near $\theta = 0$, and — near $\theta = \pi$.

[When $d < ae^2$ there are two normals from the pole besides the axis; the sign changes at the foot of each.]

Differentiate (17) twice with respect to θ, putting $R = \sqrt{\beta u^2 + \gamma}$ (positive):

$$\left.\begin{aligned} -\gamma\sin\theta &= \left(\alpha - \frac{\beta u}{R}\right)\frac{du}{d\theta} \\ -\gamma\cos\theta &= \left(\alpha - \frac{\beta u}{R}\right)\frac{d^2u}{d\theta^2} - \frac{\beta\gamma}{R^3}\left(\frac{du}{d\theta}\right)^2 = -\alpha u + R. \end{aligned}\right\} \tag{19}$$

Substituting for $\left(\frac{du}{d\theta}\right)^2$ the value obtained by squaring and adding (17) and (19), one has

$$\left(\alpha-\frac{\beta u}{R}\right)\left(\frac{d^2 u}{d\theta^2}+u\right)-\frac{\beta\gamma}{R}\cdot\frac{\gamma^2-(\alpha u-R)^2}{(\alpha R-\beta u)^2}=R-\frac{\beta u^2}{R}=\frac{\gamma}{R} \quad (20)$$

whence

$$u+\frac{d^2 u}{d\theta^2}=\frac{\gamma^2(\beta\gamma+\alpha^2-\beta)}{(\alpha R-\beta u)^2}. \quad (21)$$

Multiplication by $h^2 u^2$ gives the required law of force [if $d > ae^2$]

$$f=\frac{\varepsilon u^2}{(\alpha R-\beta u)^3}=\frac{\varepsilon\rho}{(\alpha\sqrt{\beta+\gamma\rho^2}-\beta)^3}, \quad (22)$$

where

$$\varepsilon=h^2\gamma^3\left(\beta+\frac{\alpha^2-\beta}{\gamma}\right)=h^2 a^2 e^6(1-e^2)^2. \quad (23)$$

[When $d < ae^2$, the corresponding law in any part of the ellipse is obtained from (22) by assigning to the radical a sign opposite to that specified above for the equation (17).]

The laws represented by (22) have quite different properties according as $d \gtrless ae$; though in both cases f vanishes "at infinity".

Type I, where $d > ae$. Here, since $\beta > 0$, f is everywhere real; also f is everywhere finite since the vanishing of the denominator would require $\rho^2=\rho_0^2\equiv\frac{\beta(\beta-\alpha^2)}{\alpha^2\gamma}$, a negative quantity; and f is everywhere positive [attractive], for $\beta > \alpha\sqrt{\beta+\gamma\rho^2}$ would require $\rho^2 < \rho_0^2$.

Type II, where $d < ae$. Here, since $\beta < 0$, f is imaginary within the circle $\rho^2=-\frac{\beta}{\gamma}$; also f becomes infinite at a finite distance ρ_0 [only for the law required near $\theta=0$ when $d < ae^2$, — for the others as in Type I]; and in the same case is negative [repulsive] for $\rho > \rho_0$. An exception is the very important case where $\alpha=0$: the center of force lies at the center of the ellipse and the law is that of the direct distance.

Another special case of interest is that where $d=ae^2$, the center of force being the center of curvature for the end of the major axis. The statement has

been made* that a particle can never, in a finite time, arrive at an apse, the center of whose osculating circle is at the center of force, but must approach the osculating circle asymptotically. The present example is a case to which the statement does not apply, since the time from any point (u_0, θ_0) to the apse $(u_1, 0)$ is given by

$$T = \int_{u_0}^{u_1} \frac{\phi(u)\, du}{(u - u_1)^{\frac{1}{2}}}$$

where $\phi(u_1) \neq 0, \infty$. This integral is finite by a standard test.

§ 8. *Proof of the Operation of the Newton's Law.*

A property common to the laws of both types, except in special cases, is that the force vanishes at the center of force. For, from (22), it is evident that $f = 0$ when $\rho = 0$, unless $\sqrt{\beta}\,(\pm\, a - \sqrt{\beta}) = 0$ which would require either $d = ae$ or $d = a$. The latter exception has been excluded, since in that case the center of force would lie upon the ellipse. The former exception occurs when the center of force is at a focus; and in this case, since $\beta = 0$, (22) reduces to Newton's law, as it should.

Therefore, as the vanishing of f for $\rho = 0$ contradicts hypothesis (II), the Newtonian law must operate in the given binary system.

[It is evident that the method of this section may be applied equally well to exterior points in the line of the major axis. And, again, the laws of type II, which prescribe an imaginary force in a part of the plane, might have been excluded otherwise by strengthening hypothesis (III) to require the possibility of real motion throughout the plane. Reasons for the choice of the method here adopted will be given in § 15.]

PART II. TRAJECTORIES FOR THE LAWS OF TYPE I.

§ 9. *Stability.*

Any law of types I and II admits the given ellipse as a central orbit, in which, at any instant, there are certain initial conditions. The question naturally arises as to the sort of curves that would be described if the initial conditions were different.

* Cf. Routh, *loc. cit.*, p. 288. This author probably intended, however, to have his statement understood as applying to only the special types of laws discussed in earlier pages, and not to be taken in the general form in which ennunciated.

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With the exception of the law of the direct distance, whose trajectories are well known, no law of type II permits real motion in all parts of the plane. These laws will not be discussed further. All laws of type I (including that of Newton, whose simple trajectories will not be discussed here), produce for any real initial conditions real orbits, some of whose properties will now be established.

Consider the law (22) for any chosen value of d, $(ae < d < a)$. If a particle be projected from any point in any direction, with a velocity less than a certain function of the initial distance (the "velocity from infinity"), the orbit is stable. That is, the motion is periodic, there being both a pericenter and an apocenter; and a slight change in the conditions of motion at any instant will produce only a slight change in the apsidal distances.

At $t = t_0$ let $u = u_0$, $R = R_0$; and let the velocity be v_0 and the constant of areas be h_2. Then, if $h_2 \neq 0$, the differential equation of the path is, by (22),

$$h_2^2\left(u + \frac{d^2 u}{d\theta^2}\right) = \frac{\varepsilon}{(\alpha R - \beta u)^3}. \tag{24}$$

Integration of (24) after multiplication by $2\dfrac{du}{d\theta}$ gives

$$h_2^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = k_2 + F(u) \equiv k_2 + \frac{\varepsilon u (2\alpha R - \beta u)}{\alpha^2 \gamma (\alpha R - \beta u)^2}, \tag{25}$$

where k_2 is a constant, and where the left member equals the square of the velocity. It is easily seen that the limit of $k_2 + F(u)$ as u becomes infinite is finite. Hence there is some value u_1, such that, for $u > u_1$, $h_2^2\left(\dfrac{du}{d\theta}\right)^2 < 0$. But $h_2^2\left(\dfrac{du}{d\theta}\right)^2 = \left(\dfrac{d\rho}{dt}\right)^2$. Therefore there is a circle $u = u_1$, within which the radial velocity, — and likewise the motion, — is imaginary for the given initial conditions. This establishes the existence of the pericenter.

From the initial conditions follows the equation

$$k_2 = v_0^2 - F(u_0) = v_0^2 - V_0^2, \tag{26}$$

where V_0 is the velocity acquired by a particle in falling from rest at an infinite distance to the initial point, and is real. The latter statement follows from the fact that ε, α, γ, $(\alpha R_0 - \beta u_0)$, — and hence also $2\alpha R_0 - \beta u_0$, — are positive;

so that $F(u_0) > 0$. Now it is possible to choose $v_0 < V_0$; and for any such choice $k_2 < 0$, and $v^2 < 0$ at $u = 0$ as follows from (25). Therefore $h_2^2 \left(\frac{du}{d\theta}\right)^2 < 0$ for all values of u less than some value u_2; so that the motion is imaginary outside the circle $u = u_2$. Thus the orbit has an apocenter.

Since in all central orbits included between two apsidal distances, the apsidal angle and time are constant, if the force be single-valued as required by (22), the radius vector is a periodic function of the longitude and of the time.

If $h_2 = 0$, then $\frac{d\theta}{dt} = 0$ and $\frac{d^2 r}{dt^2} = \frac{\varepsilon u^2}{(\alpha R - \beta u)^3}$. These two equations show the motion to be a periodic oscillation in the line $\theta = \theta_0$.

From (26) it follows that a small change in v_0 would affect k_2 but slightly. And, since $h_2 = v_0 \rho_0 \sin \psi_0$ (where ψ_0 is the angle between the radius vector and tangent at $t = t_0$), a small change in either v_0 or ψ_0 would produce only a small change in h_2. Hence u_1 and u_2, the values of u for which $\frac{du}{d\theta} = 0$, would change but little. The orbit is stable.

§ 10. *Other Conics as Orbits.*

Another question presents itself: is the given ellipse a solitary conic orbit under this law of force, or are there others, — in particular, other ellipses? It will be shown that there is a double infinitude of elliptic orbits, and that through every point of the plane there passes in every direction one, and only one, conic section which is a possible orbit.

In the first place if any other orbit be an ellipse, the center of force must lie upon the major axis and not between the foci; for, if it were elsewhere, the law would be other than of type I.

Hereafter let the constants of the original ellipse be denoted by the subscript 1, and the corresponding constants for any other elliptic orbit, by the same letters without subscripts. Then the equation of any such orbit is

$$\gamma \cos(\theta - \nu) = \alpha u - \sqrt{\beta u^2 + \gamma}, \tag{27}$$

ν being an arbitrary constant, the longitude of the pericenter. Also the law under consideration becomes in the new notation:

$$f = \frac{\varepsilon_1 \rho}{(\alpha_1 \sqrt{\beta_1 + \gamma_1 \rho^2} - \beta_1)^3}; \tag{22$_1$}$$

and similarly the force acting in the new ellipse is given by

$$f = \frac{\varepsilon\rho}{\alpha\sqrt{\beta+\gamma\rho^2}-\beta)^3}, \tag{28}$$

where $\varepsilon = h^2 a^2 e^2 (1-e^2)^2$, h being the new constant of areas. Hence for all values of ρ, the values of f in (22_1) and (28) are identical; from which it follows that the irrational terms in the two must be identical, and, likewise, the rational terms. The application of this principle to $\sqrt[3]{\frac{\rho}{f}}$ in (22_1) and (28) gives

$$\frac{\alpha\sqrt{\beta}}{\varepsilon^{1/3}} = \frac{\alpha_1\sqrt{\beta_1}}{\varepsilon_1^{1/3}}, \quad \frac{\alpha\sqrt{\gamma}}{\varepsilon^{1/3}} = \frac{\alpha_1\sqrt{\gamma_1}}{\varepsilon_1^{1/3}}, \quad \frac{\beta}{\varepsilon^{1/3}} = \frac{\beta_1}{\varepsilon_1^{1/3}}. \tag{29}$$

From (29) it follows immediately that

$$\frac{\alpha}{\sqrt{\gamma}} = \frac{\alpha_1}{\sqrt{\gamma_1}}, \quad \frac{\beta}{\gamma} = \frac{\beta_1}{\gamma_1}, \quad \frac{\varepsilon}{\gamma^3} = \frac{\varepsilon_1}{\gamma_1^3}. \tag{30}$$

That is, in all elliptic orbits which may be possible, the three quantities $\frac{\alpha}{\sqrt{\gamma}}$, $\frac{\beta}{\gamma}$ and $\frac{\varepsilon}{\gamma^3}$ must remain invariant. Let these be denoted by A, B and E, respectively; then $\alpha = Ae$, $\beta = Be^2$ and $\varepsilon = Ee^6$. The substitution of these values in (22_1) gives for the law of force:

$$f = \frac{E\rho}{(A\sqrt{B+\rho^2}-B)^3}, \tag{31}$$

and in (27) gives

$$e\cos(\theta-\nu) = Au - \sqrt{Bu^2+1}; \tag{32}$$

so that any possible elliptic orbit under the law (31) is represented by (32) for some value of e, — the original ellipse for $e = e_1$.

Conversely, it will be shown that for all values of e and ν the curves (32) are conics, — a double infinitude possessing certain properties because of the invariants A and B. And further there are real initial conditions, — determined by two invariants, E and another (obtained later), — for which each member of the family of conics (32) is a possible orbit under the law (31).

§ 11. *The Family of Conics.*

Consider the equation (32) for arbitrary values of e and ν. All the distinct curves which are represented by it are obtained for $0 \lesseqgtr e < \infty$ and $0 \lesseqgtr \nu \lesseqgtr 2\pi$,

the curves obtained for negative values of e being included in these. For, by changing the sign of e and adding $\pm\pi$ to ν, the equation is unchanged.

To vary ν while keeping e constant does not change the shape of a curve, but merely rotates it in the plane. It is, therefore, sufficient to consider the curves for which $\nu = 0$. This particular infinitude will hereafter be called *the restricted family*; its geometrical properties are possessed by all families in which ν has a fixed value.

If rectangular coördinates be introduced by $\cos\theta = xu$, $\sin\theta = yu$, the equation (32) with $\nu = 0$ becomes:

$$x^2(1-e^2) + 2Aex + y^2 = D, \tag{33}$$

where

$$D = A^2 - B = (a_1^2 - d_1^2)(1 - e_1^2) > 0.$$

Equation (33) represents for any value of e a conic section, whose transverse axis is in the x-axis, and whose eccentricity is e. For $e \neq 1$, the center is $\left(-\frac{Ae}{1-e^2},\ 0\right)$, and the square of the major semi-axis is $a^2 = \frac{Be^2 + D}{(1-e^2)^2}$. The apocentral distance in any ellipse is, then,

$$\frac{(Be^2 + D)^{\frac{1}{2}} + Ae}{1 - e^2} \text{ or } \frac{D}{(Be^2 + D)^{\frac{1}{2}} - Ae};$$

similarly the pericentral distance in any conic is $\frac{D}{(Be^2 + D)^{\frac{1}{2}} + Ae}$. The former steadily increases with e, having its smallest value $\sqrt{D}$ for $e = 0$, and becomes infinite as $e \doteq 1$. The latter is a decreasing function of e, having its largest value $\sqrt{D}$ for $e = 0$, and approaches zero as $e \doteq \infty$. Hence every conic of the general family has its pericenter within, and its apocenter without, the circle of the family, $\rho^2 = D$.

Various geometrical properties, belonging chiefly to the restricted family, are readily seen. As neither of the invariants A and B is so well suited to interpretation as two of their combinations, viz., D and $\frac{A}{\sqrt{D}}$, these latter two will be used.

I. THE FIRST INVARIANT, D. *All conics of the family with collinear axes pass through two fixed points;* viz., the points where the circle $\rho^2 = D$ meets the

perpendicular to the axial line through the origin. For equation (32) with ν fixed is satisfied by $\rho = \sqrt{D}$, $\theta = \nu \pm \frac{\pi}{2}$, whatever be the value of e. [Another proof rests upon the fact that all members of the restricted family pass through $(0, \pm \sqrt{D})$.]

It is possible to define new families of conics related to the family (32) in which there is a simple interpretation of this and the following invariants. If a new conic be described, having the same eccentricity as some given conic of the restricted family, and having as its transverse axis the segment between the center of the given conic and the pole with respect to the given conic of the line joining the fixed points $(0, \pm \sqrt{D})$, then *this new conic passes through the same two fixed points, whatever the value of e.** To establish the property it is sufficient to note the coördinates of the vertices of the new conic,

$$\left(-\frac{Ae}{1-e^2},\ 0\right) \text{ and } \left(\frac{D}{Ae},\ 0\right).$$

Evidently the auxiliary circle of the new conic cuts the Y-axis where $Y^2 = \frac{D}{1-e^2}$, Y being the mean proportional between the two segments of the axis. Let y be the ordinate in the new conic at $x = 0$. Now, in any ellipse, $y^2/Y^2 = b^2/a^2$; and, in any hyperpola, $y^2/Y^2 = -b^2/a^2$. In either case, $y^2 = Y^2(1-e^2) = D$.

II. THE SECOND INVARIANT, $\frac{A}{\sqrt{D}}$. *In all conics of the restricted family, the slope at either fixed point bears to the slope at the end of the latus rectum a ratio which is constant.* For the latter slope is $\pm e$, while the former is seen from (33) to be $\pm \frac{Ae}{\sqrt{D}}$. The ratio is $\pm \frac{A}{\sqrt{D}}$, which is constant.

If a new conic be described, having the same center and eccentricity as some given conic of the restricted family, and having a focus at the origin, the new conic passes through two fixed points whatever the value of e, so that *its latus rectum bears a constant ratio to the segment between the fixed points of the given family.* For, the major semi-axis of the new conic being $\frac{d}{e}$, its latus rectum is

* The members of one family do not belong to the other (except for a single value of e). For in each family there is but one conic with a given eccentricity, and corresponding conics have axes of different lengths.

$\frac{2d(1-e^2)}{e}$ or $2A$; and the distance between the fixed points $(0, \pm\sqrt{D})$ being $2\sqrt{D}$, the ratio is $\frac{A}{\sqrt{D}}$. Again, since the constant latus rectum lies always in the Y-axis, all the new conics pass through $(0, \pm A)$.

III. AN ALTERNATIVE INVARIANT, B. *The family defined in II intersect the tangents at the vertices of corresponding members of the given family on two straight lines, $y^2 = B$; and conversely, on $y^2 = -B$.* These statements may be proved analytically by using the equations of the tangents, or geometrically by observing that the segments into which the major axis of the new conic is divided by the vertex of the old are $\frac{d}{e} - a$ and $\frac{d}{e} + a$. It follows that the tangent is cut by the auxiliary circle of the new conic where $y^2 = \frac{d^2 - a^2e^2}{e^2}$, and hence by the conic where $y^2 = \frac{(d^2 - a^2e^2)(1-e^2)}{e^2} = B$. Similarly for the converse.

IV. *Any two conics of the general family intersect, so that no one lies wholly within any other.* For let the values of ν and e in the two curves be respectively ν_1, e_1 and ν_2, e_2; and let the radii vectores be denoted by ρ_1 and ρ_2. Suppose $e_2 > e_1$. Then, by the remark above concerning the pericentral distance, $a_2 - d_2 < a_1 - d_1$; and hence, when $\theta = \nu_2$, $\rho_2 < \rho_1$, for $a_1 - d_1 \overline{\gtrless} \rho_1$.

Case I. When $e_2 < 1$, both orbits have apocenters, and $a_2 + d_2 > a_1 + d_1$. Hence, for $\theta = \nu_2 + \pi$, $\rho_2 > \rho_1$; and, for some value of θ between ν_2 and $\nu_2 + \pi$, the two ellipses must intersect.

Case II. When $e_2 \overline{\gtrless} 1 > e_1$, the ellipse has an apocenter, while the other conic extends to infinity. Hence there are values of θ where $\rho_2 > \rho_1$, so that the curves cut.

Case III. When $e_1 \overline{\gtrless} 1$, there exist values of θ where $\rho_2 > \rho_1$. For the value of $\theta - \nu$ where any conic of (32) goes to infinity is $\phi = \text{arc}\cos\left(-\frac{1}{e}\right)$, showing that $\phi > \frac{\pi}{2}$ and ϕ is a decreasing function of e. Now ρ is finite from $\theta = \nu - \phi$ to $\theta = \nu + \phi$; and since $\phi_1 > \phi_2 > \frac{\pi}{2}$, the range of values of θ for which ρ_2 is finite is smaller than that for which ρ_1 is finite, but is greater than π. Hence at least one of the radii vectores where ρ_2 becomes infinite lies in the region where ρ_1 is finite. Near this, $\rho_2 > \rho_1$.

V. *Through every point there passes in every direction one, and only one, conic of the family* (32). For, let the point be (u_0, θ_0), and let the given direction make with the radius vector to that point the angle ψ_0. Then, in the required curve, $\tan\psi_0 = -u_0\left(\frac{d\theta}{du}\right)_0$, where from (32):

$$-\frac{d\theta}{du} = \frac{A\sqrt{Bu^2+1} - Bu}{\sqrt{Bu^2+1}\,[e^2 - (Au - \sqrt{Bu^2+1})^2]^{\frac{1}{2}}}. \tag{34}$$

Hence if any member of the family is to have $\psi = \psi_0$ at $u = u_0$, its eccentricity must have the positive (or zero) value given by

$$e_2^2 = (Au_0 - \sqrt{Bu_0^2+1})^2 + \left(A - \frac{Bu_0}{\sqrt{Bu_0^2+1}}\right)^2 u_0^2 \cot^2\psi_0. \tag{35}$$

Evidently $e_2^2 \geqq 0$, so that there is a real positive (or zero) solution. The substitution of $e = e_2$, $\theta = \theta_0$ and $u = u_0$ in (32) determines ν so that the conic will pass through the given point. The equations determine both e_2 and ν uniquely, subject to the original restriction.

COROLLARY. *Every point in the plane is an apse in some conic of the family.* For then $\psi_0 = \frac{\pi}{2}$, and equations (35) and (32) become $e_2 = \pm(Au_0 - \sqrt{Bu_0^2+1})$, and $\cos(\theta_0 - \nu) = \pm 1$, where the upper or lower sign is to be taken according as $A^2 \gtrless B + \rho_0^2$, — or $\rho_0^2 \lessgtr D$. Further, if $\rho_0^2 < D$, $\theta_0 = \nu$; while, if $\rho_0^2 > D$, $\theta_0 = \nu \pm \pi$. That is, any given point within the circle $\rho^2 = D$ is the pericenter in some conic of the family; and any point without that circle is the apocenter in some conic of the family. [This corrollary establishes the converse of a property obtained above.]

§ 12. *The Initial Conditions.*

It has been shown that, if the law (31) admits as trajectories any ellipses other than the original orbit, those ellipses must be members of the family of conics just discussed; and it is further necessary that the constants of areas have values determined as follows:

THE THIRD INVARIANT, E, or $h^2 a^2 (1 - e^2)^2$. *The constant of areas must be proportional to the curvature at an apse.* For the radius of curvature at an apse is $a(1-e^2)$, so that the constant of areas h divided by the curvature is invariant.

Several questions now require consideration.

I. *Is an arbitrarily chosen constant of areas h_2 admissible for some conic of the family?* If so, then between h_2 and the required eccentricity e_2 must exist the relation $E = h_2^2\left(\beta_2 + \frac{\alpha_1^2 - \beta_2}{\gamma_2}\right) = h_2^2(B e_2^2 + D)$, which requires

$$h_2^2 B e_2^2 = E - h_2^2 D. \tag{36}$$

Equation (36) defines a unique positive number e_2, provided simply that $h_2^2 < \frac{E}{D}$, the value of h^2 in the circle of the family. [If $h_2 = 0$, $e_2 = \infty$, the curve being a straight line, as is indeed evident from other considerations.]

This result does not show, however, that there is an upper limit to the initial velocity of projection. For $h = v\rho \sin\psi$ constantly; and therefore arbitrarily large values of v yield admissible values of h, if only the initial distance or the angle ψ be sufficiently small.

II. *For an arbitrarily chosen constant of areas h_2*, less than the limit mentioned above, *what further restriction must be put upon the initial conditions in order that the trajectory be a conic?* Let the values of ρ, u, θ, v and ψ at the initial point be designated by the subscript zero. Now in order that the trajectory be a conic, e must have the value e_2, defined by (36).

Hence, in the first place, ρ_0 must lie between $a_2 - d_2$ and $a_2 + d_2$ [or ∞ if $e_2 \gtreqless 1$]. Further ψ_0 must equal the value of ψ at the distance ρ_0 in the conic for which $e = e_2$. And, finally, v_0 must be given by $v_0 = h_2 u_0 \csc\psi_0$.

To formulate the second of these conditions, consider equation (25). Since, in any curve, $\tan\psi = -u\frac{d\theta}{du}$, or $u^2\csc^2\psi = u^2 + \left(\frac{du}{d\theta}\right)^2$, the value of $u_0^2\csc^2\psi_0$ in the orbit (25) is, for arbitrary conditions of projection,

$$u_0^2\csc^2\psi_0 = \frac{k_2}{h_2^2} + \frac{E u_0(2A\sqrt{B u_0^2 + 1} - B u_0)}{A^2 h_2^2 (A\sqrt{B u_0^2 + 1} - B u_0)^2}. \tag{37}$$

And in the conic for $e = e_2$, the value at $u = u_0$ is

$$u_0^2\csc^2\psi = u_0^2 + \frac{(B u_0^2 + 1)\left[e_2^2 - (A u_0 - \sqrt{B u_0^2 + 1})^2\right]}{(A\sqrt{B u_0^2 + 1} - B u_0)^2}. \tag{38}$$

Equating the right members of (37) and (38), and observing that

$$E = h_2^2\left[B(e_2^2 - 1) + A^2\right],$$

a condition on $\frac{k_2}{h_2^2}$ is obtained. The irrational parts must be identical, likewise the rational parts:

$$\begin{aligned}\frac{k_2}{h_2^2}(-2A^3Bu_0)\sqrt{Bu_0^2+1} &= -2ABu_0(e_2^2-1)\sqrt{Bu_0^2+1}\\ \frac{k_2}{h_2^2}A^2[A^2+u_0^2(A^2B+B^2)] &= (e_2^2-1)[A^2+u_0^2(A^2B+B^2)].\end{aligned} \tag{39}$$

Each equation requires that $\frac{k_2}{h_2^2}=\frac{e_2^2-1}{A^2}$; that is, for all orbits of the family, $\frac{h^2(e^2-1)}{k}$ is *a fourth invariant*, restricting the initial conditions.

For a conic orbit with the constant of areas h_2, the velocity of projection at the distance ρ_0 must be v_0, where

$$v_0^2 = k_2 + F(u_0) = h_2^2\,\frac{e_2^2-1}{A^2} + F(u_0); \tag{40}$$

and ψ_0 is given by $h_2 = v_0\rho_0 \sin\psi_0$. Or, reversing the order, ψ_0 may be obtained from (38), and then v_0 from either of the two equations last written.

III. *To determine the velocity, with which a particle projected at any point in any direction will describe a conic.* There is one, and only one, conic through the given point in the given direction. Since u_0 and ψ_0 are known, (35) determines e_2, real and positive; and (36) then determines h_2. Finally v_0 is given by (40) or by $v_0 = h_2 u_0 \csc\psi_0$. It is evident that there is one, and only one, real positive value of v_0 for which the orbit is a conic. The fact that there are initial velocities for which the orbits are other curves than conics, is important in a later section.

§ 13. *The Apsidal Angle in the Non-Conic Orbits.*

The question as to whether "the line of apsides advances or recedes," — that is, as to whether the apsidal angle Θ is greater than or less than π, — is a matter of interest. The expression for the force in laws of type I may be written, $f = u^2 P(u)$, where

$$P(u) \equiv \frac{E}{(AR-Bu)^3}. \tag{41}$$

Now it has been shown elsewhere* that, if $P'(u) > 0$ for $u_1 > u > u_2$, then

* F. L. Griffin: On the Apsidal Angle in Central Orbits, *Bulletin of the American Mathematical Society*, Vol. XIV (1907), pp. 6–16.

$\Theta > \pi$ in all orbits lying wholly between the circles $u = u_1$ and $u = u_2$; and conversely. Similarly, if $P'(u) < 0$, $\Theta < \pi$. And further, if $P'(u) < 0$, and decreases with u or is constant, then in any family of orbits, one of whose apsidal distances is constant, Θ increases with the second apsidal distance. Analogous statements apply in the opposite case.

For type I,

$$P'(u) = \frac{-3EB}{R(AR - Bu)^4}(Au - R). \tag{42}$$

Hence $P'(u) > 0$ if $R > Au$; that is, if $\rho^2 > D$; while $P'(u) < 0$ if $\rho^2 < D$. Consequently in all the orbits which lie wholly within the circle $\rho^2 = D$, $\Theta < \pi$; and in all orbits wholly without the circle $\rho^2 = D$, $\Theta > \pi$. [It is interesting to compare this result with the fact that in the conic orbits $\Theta = \pi$, and to note that a new proof is thus obtained for the earlier statement that every conic orbit admitted by the law (31) has its pericenter within and its apocenter without the circle $\rho^2 = D$.]

Moreover, since $P'\left(\frac{1}{\sqrt{D}}\right) = 0$ and $P'(u) < 0$ when $u < \frac{1}{\sqrt{D}}$, there exists a region within the circle $\rho^2 = D$ and without another concentric circle, throughout which Θ decreases as the apsidal distances in the orbits decrease The corresponding question for orbits outside $\rho^2 = D$ requires more extended computation.

§ 14. *A New Proof with Classic Hypotheses.*

From the foregoing discussion, it is possible to obtain a new proof of the operation of the Newtonian law in the binary systems, based upon the hypotheses mentioned in § 1 in connection with the classic theorem of Halphen and Darboux.

The classic hypotheses imply those of the present paper; for (I) and (II) are common, and, if every orbit is a conic, then the law according to which the force varies (not merely in one orbit but throughout the plane) is such as to permit real motion everywhere. And, since the operation of Newton's law follows from the new hypotheses, it follows also from the old ones.

Again, a new proof is possible with the assumptions of Halphen and Darboux modified as suggested in § 1. For the second theorem of Bertrand referred to may be applied, with assumption (III) and the fact that at least one conic orbit is an ellipse, to show that the force is central. Consider that ellipse; the center of force may lie only upon the major axis. For, if it were elsewhere, the use

of (I) would furnish a law, for which some initial conditions would give rise to imaginary orbits, [§ 6], in violation of (III). Moreover, if the center of force were any point of the major axis save the center [excluded by assumption] or a focus, the corresponding law would admit imaginary orbits (if of type II) or real non-conic orbits (if of type I), [§ 12], again in violation of (III). Thus, Newton's law must operate.

§ 15. *Possible Modifications in Hypotheses.*

(1) As observed above, laws of type II might be excluded by strengthening hypothesis (III) to require the possibility of real motion, not merely sufficiently far from the center of force, but throughout the plane, an increase in severity which may seem on first consideration to be extremely small.

It is true, however, that the hypothesis in this revised form would exclude so important a law as that according to which an oblate spheroid attracts an exterior particle in its equatorial plane; for, if the latter law be required to operate within the spheroid, the force is imaginary within a certain interior circle.* But it is well to observe that even this exclusion would not be excessive in the present problem, inasmuch as the latter law admits as orbits no conics except circles.

Nevertheless, if the attracting body be regarded as having size, the severity of the revised hypothesis is greatly enhanced, as it would exclude the operation at exterior points of any law which, if operating also at interior points, would prescribe imaginary values of the force within the attracting body, even though all (physically possible) real initial conditions give rise to real orbits. If modified to require only the reality of the force everywhere outside the attracting body, the hypothesis would not suffice to exclude laws of type II for which $ae > d > ae^3$, and would suffice to exclude those where $d < ae^3$ only when strengthened by the assumption that the bodies do not collide. [These latter statements rest upon the fact that, if ρ_1 denote the value of ρ for which the imaginary radical vanishes, $\rho_1 < a - d$ (the least radius vector in the orbit) if $ae > d > ae^3$, and $\rho_1 =$ the minimum normal if $d < ae^3$.] When $ae > d > ae^3$, however, the center of force would at least have to be sufficiently near the focus to have the principal star contain the focus; for the inequality $\rho_1 > ae - d$ follows from the known inequality $0 > (d - ae)(d + ae - 2ae^3)$.

* For the law in question see Moulton. *loc. cit.*, p. 123, or paper cited in *Bulletin*, p. 13.

(2) Again, it has been noted that all exterior points, and all points on the ellipse except the ends of the major axis, could be excluded without the aid of the reference cited in § 1. These two, and all other, points of the curve might be excluded by formulating as an assumption for material points the physical principle of impenetrability, or by otherwise prohibiting collision. This change would, however, involve an altogether unnecessary (even if mild) addition to the assumptions of the paper.

(3) Finally, in any non-rigid body tides would be formed with a resulting difference in the attracting force, which, considered as a perturbation, would in general forbid the constancy of $\frac{h^2(e^2-1)}{k}$ (the fourth invariant). Doubtless a substitute for hypothesis (II) can be framed, based upon some such requirement as that, under tidal disturbance the force shall cause the velocity at any instant to be such as to permit the description of a conic as the subsequent orbit were tidal disturbance to disappear. In addition, it would be necessary to require that the center of force be not at the center of the orbit, in order to exclude the law of the direct distance which compels an elliptic orbit for all initial conditions.

WILLIAMS COLLEGE, *September*, 1908.

Lösung des Lehmer'schen Problems.

Von Edmund Landau in Berlin.

Einleitung.

Es ist mir durch Kombination meiner älteren Methoden mit einigen neueren Hilfsmitteln gelungen, ein seit Jahren angestrebtes Ziel zu erreichen. Ich kann jetzt ein von Herrn Lehmer[1] im Jahre 1900 gestelltes Problem in voller Allgemeinheit lösen, von dem ich drei wichtige Spezialfälle (für welche die alten Methoden nicht ausreichten) schon in einer früheren Arbeit[2] erledigt hatte.

Es seien a, b zwei teilerfremde positive ganze Zahlen;[3] $\Theta(n)$ bedeute 1 oder 0, je nachdem alle Primfaktoren von n die Form $am+b$ haben oder nicht; $\nu(n)$ sei die Anzahl der verschiedenen Primfaktoren von n. Herr Lehmer fragte zunächst, ob

$$\lim_{x=\infty} \frac{\sum_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{x} \tag{1}$$

stets existiert. Zu dieser Frage gelangt er dadurch, dass in den drei Fällen 1) $a=3$, $b=1$, 2) $a=4$, $b=1$, 3) $a=6$, $b=1$ die Theorie der quadratischen Formen den Nachweis der Existenz des Grenzwertes (1) liefert; Herr Lehmer führte dies genauer aus und fand auch, dass jener Wert jedesmal >0 ist. Er vermutete nun allgemein, dass der Grenzwert existiert und >0 ist.

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[1] "Asymptotic Evaluation of certain Totient Sums" [American Journal of Mathematics, Bd. XXII (1900), S. 293–335], vergl. insbesondere den Anfang (S. 293) und den Schluss (S. 334–335) dieser Arbeit, welche auch separatim als Dissertation der University of Chicago erschienen ist und das betreffende Problem für gewisse elementar angreifbare Fälle löst.

[2] "Bemerkungen zu Herrn D. N. Lehmer's Abhandlung in Bd. 22 dieses Journals, S. 293–335" [American Journal of Mathematics, Bd. XXVI (1904), S. 209–222]. Ein anderer Beweis meiner damaligen Resultate steht in meiner späteren Arbeit "Über die Multiplikation Dirichlet'scher Reihen" [*Rendiconti del Circolo Matematico di Palermo*, Bd. XXIV (2. Semester 1907), S. 81–160], S. 143–145 (§ 13, "Neuer Beweis einer Vermutung von Herrn Lehmer").

[3] Es ist keine Einschränkung der Allgemeinheit, $b \leq a$ anzunehmen.

Darauf wies ich a. a. O. nach: 1) Wenn der Grenzwert existiert, kann er nur für $\phi(a) = 2$, d. h. für $a = 3$, $a = 4$ und $a = 6$ von 0 verschieden sein. 2) Für $a = 3$, $a = 4$ und $a = 6$, d. h. für die drei Lehmer'schen Progressionen $3m + 1$, $4m + 1$ und $6m + 1$, sowie für die drei neuen Progressionen $3m + 2$, $4m + 3$ und $6m + 5$ existiert der Grenzwert und ist > 0.

Heute kann ich nun beweisen, dass[4] für $\phi(a) > 2$, d. h. für $a = 5$ und alle $a \geq 7$ nebst jedem zugehörigen zu a teilerfremden b der Grenzwert

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)} \Theta(n)}{\dfrac{x}{(\log x)^{1-\frac{2}{\phi(a)}}}}$$

vorhanden und > 0 ist; daraus folgt insbesondere

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)} \Theta(n)}{x} = 0.$$

Herr Lehmer stellte aber noch ein viel allgemeineres Problem. Es seien λ unter den $h = \phi(a)$ zu a teilerfremden Restklassen gegeben:[5]

$$am + b_1, \quad am + b_2, \quad \ldots, \quad am + b_\lambda \quad (1 \leq \lambda \leq h).$$

Es sei $\Theta(n) = 1$ oder $= 0$, je nachdem alle Primfaktoren von n irgend einer dieser λ Progressionen angehören oder nicht. Herr Lehmer fragte, ob

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)} \Theta(n)}{x}$$

stets existiert. Er konnte nämlich mit Hilfe der Theorie der quadratischen Formen beweisen, dass für gewisse unendlich viele Fälle, in denen jedesmal $\lambda = \frac{h}{2}$ ist, der Grenzwert existiert und > 0 ist.

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[4] Für $\phi(a) = 1$, d. h. $a = 1$, $b = 1$ ist bekanntlich

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)} \Theta(n)}{x \log x} = \lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}}{x \log x}$$

vorhanden und > 0; vergl. z. B. die Reproduktion dieser klassischen Dirichlet-Mertens-Lipschitz'schen Untersuchung bei Herrn Bachmann, "Die analytische Zahlentheorie", Leipzig, 1894, S. 430–436 und S. 450–453.

[5] Der Fall $\lambda = h$ liesse sich übrigens unmittelbar auf das in Anm. 4 erwähnte klassische Problem zurückführen, d. h. elementar erledigen; $\Theta(n)$ ist ja alsdann $= 1$ für alle zu a teilerfremden Zahlen, sonst $= 0$. $\lambda = 1$ ist der oben im Text besprochene Fall. Also ist $1 < \lambda < h$ das Neue.

Ich bin heute imstande zu beweisen, dass unter allen Umständen

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{\frac{x}{(\log x)^{1-\frac{2\lambda}{h}}}}$$

existiert und >0 ist. Damit ergibt sich insbesondere: Für $\lambda<\frac{h}{2}$ ist

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{x} = 0;$$

für $\lambda=\frac{h}{2}$ ist stets (nicht nur in den Lehmer'schen Spezialfällen)

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{x}$$

vorhanden und >0; für $\lambda>\frac{h}{2}$ wächst der Quotient

$$\frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{x}$$

über alle Grenzen.

Die vorliegende Arbeit soll alle obigen Behauptungen beweisen und braucht sich nur mit der allgemeinsten unter ihnen,

$$\lim_{x=\infty} \frac{\sum\limits_{n=1}^{x} 2^{\nu(n)}\Theta(n)}{\frac{x}{(\log x)^{1-\frac{2\lambda}{h}}}} > 0,$$

zu beschäftigen.

Um das Wesentliche besonders hervortreten zu lassen, beschränke ich mich darauf, die Existenz dieses positiven Grenzwertes zu beweisen. Der Leser wird ohne Mühe erkennen, dass die angewandte Methode auch gestattet, über die Geschwindigkeit der Annäherung des Quotienten an seinen Grenzwert Genaueres auszusagen.

§ 1.

Es seien $\chi_1(n), \ldots, \chi_h(n)$ die $h = \phi(a)$ Charaktere der Gruppe der zu a teilerfremden Restklassen, davon $\chi_1(n)$ der Hauptcharakter. Es seien $L_1(s), \ldots, L_h(s)$ die für $\Re(s) > 1$ ($\varkappa = 1$) bezw. für $\Re(s) > 0$ ($\varkappa = 2, \ldots, h$) durch die Gleichungen

$$L_\varkappa(s) = \sum_{n=1}^{\infty} \frac{\chi_\varkappa(n)}{n^s}$$

definierten analytischen Funktionen von $s = \sigma + ti$. Über diese Funktionen setze ich folgende in einer früheren Arbeit[6] bewiesenen Tatsachen voraus:

1) Für $\sigma \geqq 1 - \frac{1}{\log t}$ ist gleichmässig[7]

$$L_\varkappa(s) = O(\log t). \tag{2}$$

2) Es gibt ein $\beta > 0$ derart, dass für $t \geqq 3$, $\sigma \geqq 1 - \frac{1}{\log^\beta t}$, $\varkappa = 1, \ldots, h$

$$L_\varkappa(s) \neq 0$$

ist.

3) $L_1(s)$ hat im Punkte $s = 1$ einen Pol erster Ordnung; $L_1(s)$ ist sonst für $\sigma \geqq 1$ regulär und $\neq 0$. Die übrigen Funktionen $L_\varkappa(s)$ ($\varkappa = 2, \ldots, h$) sind für $\sigma \geqq 1$ (einschliesslich des Punktes $s = 1$) regulär und $\neq 0$.

4) Für $\sigma > 1$ ist

$$L_\varkappa(s) = \prod_p \frac{1}{1 - \frac{\chi_\varkappa(p)}{p^s}}.$$

1) und 2) besagen aus Symmetriegründen,[8] dass für $\sigma \geqq 1 - \frac{1}{\log t}$

$$L_\varkappa(\sigma - ti) = O(\log t)$$

und für $t \leqq -3$, $\sigma \geqq 1 - \frac{1}{\log^\beta(-t)}$, $\varkappa = 1, \ldots, h$

$$L_\varkappa(s) \neq 0$$

ist.

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[6] "Über die Primzahlen einer arithmetischen Progression" [*Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse*, Bd. CXII, Abt. IIa (1903), S. 493–535]; S. 508, 509, 512, 519.

[7] d. h. bei gegebenem a unabhängig von σ und natürlich (da $\varkappa$ nur endlich viele Werte hat) auch von $\varkappa$.

[8] Wenn $\chi_{\varkappa'}(s)$ den zu $\chi_\varkappa(s)$ konjugierten Charakter bezeichnet, ist $L_\varkappa(\sigma + ti)$ zu $L_{\varkappa'}(\sigma - ti)$ konjugiert, also

$$|L_\varkappa(\sigma + ti)| = |L_{\varkappa'}(\sigma - ti)|.$$

Ich erinnere ferner an den von Herrn Carathéodory herrührenden Satz:[9] Die analytische Funktion $F(s)$ sei für $|s-s_0| \leqq r$ regulär; es sei A das Maximum von $\Re F(s)$ für $|s-s_0| = r$ und $0 < \rho < r$. Dann ist für $|s-s_0| \leqq \rho$

$$|F(s)| \leqq |\Im F(s_0)| + |\Re F(s_0)| \frac{r+\rho}{r-\rho} + 2A\frac{\rho}{r-\rho}. \tag{3}$$

Diesen Satz wende ich folgendermassen an. Es sei

$$F(s) = F_\kappa(s) = \log L_\kappa(s)$$

die für $\sigma > 1$ durch

$$F_\kappa(s) = \sum_p \left(\frac{\chi_\kappa(p)}{p^s} + \frac{\chi_\kappa(p^2)}{2p^{2s}} + \cdots \right)$$

definierte analytische Funktion, welche bei Fortsetzung in das Gebiet $t \geqq 3$, $\sigma \geqq 1 - \frac{1}{\log^\beta t}$ auch dort regulär ist. Es werde

$$s_0 = 1 + \frac{1}{\log^\beta t} + ti, \quad r = \frac{3}{2\log^\beta t}$$

gesetzt und $t \geqq t_0$ angenommen, wo die Konstante t_0 so gewählt ist, dass alle Punkte $u+vi$ des Kreises $|s-s_0| \leqq r$ dem Gebiet

$$v \geqq 3, \quad u \geqq 1 - \frac{1}{\log^\beta v} \tag{4}$$

angehören. t_0 kann so bestimmt werden, da

$$t - \frac{3}{2\log^\beta t} \leqq v \leqq t + \frac{3}{2\log^\beta t}, \tag{5}$$

$$u \geqq 1 - \frac{1}{2\log^\beta t}$$

ist und für alle hinreichend grossen t

$$t - \frac{3}{2\log^\beta t} \geqq 3,$$

$$1 - \frac{1}{\log^\beta\left(t + \frac{3}{2\log^\beta t}\right)} \leqq 1 - \frac{1}{2\log^\beta t}$$

ist. Für $t \geqq t_0$ ist also $F(s)$ im Kreise $|s-s_0| \leqq r$ regulär, also der Satz von

[9] Vergl. meine Arbeit "Beiträge zur analytischen Zahlentheorie" [*Rendiconti del Circolo Matematico di Palermo*, Bd. XXVI (2. Semester 1908), S. 169–302], S. 191–192, wo der Satz im Anschluss an eine schriftliche Mitteilung von Herrn Carathéodory bewiesen ist. Ich verstehe für $z = u + vi$ unter $\Re(z)$ den reellen Teil u, unter $\Im(z)$ den Koeffizienten v von i.

Herrn Carathéodory anwendbar, und zwar ist nach (2), (4) und (5) die in ihm vorkommende Zahl

$$A = \underset{|s-s_0|=r}{\mathrm{Max.}}\ \Re \log L_\kappa(s) = \underset{|s-s_0|=r}{\mathrm{Max.}} \log |L_\kappa(s)| < O\left(\log\log\left(t + \frac{3}{2\log^\beta t}\right)\right)$$
$$= O(\log\log t). \tag{6}$$

Ferner ist

$$|F_\kappa(s_0)| = |\log L_\kappa(s_0)| = \left|\sum_p \left(\frac{\chi_\kappa(p)}{p^{s_0}} + \frac{\chi_\kappa(p^2)}{2p^{2s_0}} + \cdots\right)\right|$$
$$\leqq \sum_p \left(\frac{1}{p^{\Re(s_0)}} + \frac{1}{2p^{2\Re(s_0)}} + \cdots\right) = \log\zeta(\Re(s_0)) = \log\zeta\left(1 + \frac{1}{\log^\beta t}\right)$$
$$= O(\log\log t), \tag{7}$$

da

$$1 < \zeta\left(1 + \frac{1}{\log^\beta t}\right) = O(\log^\beta t)$$

ist. Wenn

$$\rho = \frac{5}{4\log^\beta t}$$

gesetzt wird, ist

$$0 < \rho < r$$

und

$$\frac{r+\rho}{r-\rho} = \frac{\frac{3}{2}+\frac{5}{4}}{\frac{3}{2}-\frac{5}{4}} = 11, \quad \frac{2\rho}{r-\rho} = \frac{\frac{5}{2}}{\frac{3}{2}-\frac{5}{4}} = 10.$$

Die Formel (3) liefert also nach (6) und (7) für

$$\left| s - \left(1 + \frac{1}{\log^\beta t} + ti\right)\right| \leqq \frac{5}{4\log^\beta t},$$

also speziell für

$$s = \sigma + ti, \quad 1 - \frac{1}{4}\frac{1}{\log^\beta t} \leqq \sigma \leqq 1 + \frac{1}{\log^\beta t}$$

$$|F_\kappa(s)| = |\log L_\kappa(s)| = O(\log\log t).$$

Für $\sigma \geqq 1 + \frac{1}{\log^\beta t}$ ist jedenfalls

$$|F_\kappa(s)| = |\log L_\kappa(s)| \leqq \log\zeta\left(1 + \frac{1}{\log^\beta t}\right) = O(\log\log t).$$

Folglich ist für $\sigma \geqq 1 - \frac{1}{4} \frac{1}{\log^\beta t}$

$$\log L_\kappa(s) = O(\log\log t),$$

also nicht nur, wie bekannt,[10]

$$\Re \log L_\kappa(s) = O(\log\log t),$$

sondern auch

$$\Im \log L_\kappa(s) = O(\log\log t).$$

Aus Symmetriegründen ist ebenso für $t \leqq -3$, $\sigma \geqq 1 - \frac{1}{4} \frac{1}{\log^\beta(-t)}$

$$|\log L_\kappa(s)| < B \log\log |t|,$$

wo B konstant ist.

Ich habe also bewiesen:[11]

Es gibt eine absolute Konstante c derart, dass in dem durch den geraden Schnitt von $1 - \frac{1}{\log^c 3}$ *bis* 1 *aufgetrennten Gebiete*

$$\left\{ \begin{array}{ll} \sigma \geqq 1 - \frac{1}{\log^c t} & \textit{für } t \geqq 3, \\ \sigma \geqq 1 - \frac{1}{\log^c 3} & \textit{für } 3 \geqq t \geqq -3, \\ \sigma \geqq 1 - \frac{1}{\log^c(-t)} & \textit{für } t \leqq -3 \end{array} \right.$$

jede der Funktionen $F_\kappa(s) = \log L_\kappa(s)$ *regulär ist und für* $|t| \geqq 3$, $\sigma \geqq 1 - \frac{1}{\log^c |t|}$ *die Ungleichung*

$$|F_\kappa(s)| < c \log\log |t|$$

erfüllt.

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[10] Dass in obigem Gebiet

$$\frac{1}{L_\kappa(s)} = O(\log^\beta t),$$

d. h.

$$\Re \log L_\kappa(s) = O(\log\log t)$$

ist, hätte ich zwar auch aus meiner in Anm. 6 zitierten Arbeit (S. 521) als bekannt voraussetzen können; aber die vorangehenden Entwickelungen waren für den neuen Hilfssatz

$$\Im \log L_\kappa(s) = O(\log\log t)$$

nötig.

[11] Wenn c hinreichend gross gewählt wird, liegt ja im Gebiet $-3 \leqq t \leqq 3$, $1 - \frac{1}{\log^c 3} \leqq \sigma \leqq 1$ keine Nullstelle eines $L_\kappa(s)$. c sei für einen späteren Zweck gleich so gross, dass $1 - \frac{1}{\log^c 3} > \frac{3}{4}$ ist.

Übrigens ist für $\varkappa = 2, \ldots, h$ der Schnitt unnötig; für $\varkappa = 1$ ist die Funktion auf ihm (exkl. $s = 1$) regulär, aber an beiden Ufern um $2\pi i$ verschieden.

§ 2.

Es werde

$$c_n = 2^{\nu(n)}$$

oder

$$c_n = 0$$

gesetzt, je nachdem alle $\nu(n)$ Primfaktoren von n einer bestimmten Progression $am + b$ (wo $(a, b) = 1$ ist) angehören oder nicht. Es sei $f(s)$ die für $\sigma > 1$ durch

$$f(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} = \prod_q \left(1 + \frac{2}{q^s} + \frac{2}{q^{2s}} + \cdots\right) = \prod_q \frac{1 + \frac{1}{q^s}}{1 - \frac{1}{q^s}} = \prod_q \frac{1}{\left(1 - \frac{1}{q^s}\right)^2} \prod_q \left(1 - \frac{1}{q^{2s}}\right),$$

wo q die Primzahlen von der Form $am + b$ durchläuft, definierte Funktion. Die für $s > 1$ reell definierte Funktion $\log f(s)$ ist für $\sigma > 1$ eindeutig; für $\sigma > 1$ ist

$$\begin{aligned}
\log f(s) &= -2 \sum_q \log\left(1 - \frac{1}{q^s}\right) + \sum_q \log\left(1 - \frac{1}{q^{2s}}\right) \\
&= 2 \sum_q \frac{1}{q^s} + \sum_q \left(\frac{2}{2q^{2s}} + \frac{2}{3q^{3s}} + \cdots\right) - \sum_q \left(\frac{1}{q^{2s}} + \frac{1}{2q^{4s}} + \cdots\right) \\
&= 2 \sum_q \frac{1}{q^s} + g_1(s),
\end{aligned}$$

wo $g_1(s)$, desgl. in der Folge $g_2(s)$, $g_3(s)$, $\ldots$, eine für $\sigma > \frac{1}{2}$ absolut konvergente Dirichlet'sche Reihe ist, wo also für $\sigma > \frac{3}{4}$ gleichmässig

$$g(s) = g(\sigma + ti) = O(1)$$

ist.

Andererseits ist für $\sigma > 1$

$$\log L_\varkappa(s) = \sum_p \frac{\chi_\varkappa(p)}{p^s} + g_2(s),$$

also, wenn l die zu b inverse Restklasse modulo a repräsentiert,

$$\sum_{\varkappa=1}^{h} \chi_\varkappa(l) \log L_\varkappa(s) = \sum_{\varkappa=1}^{h} \chi_\varkappa(l) \sum_p \frac{\chi_\varkappa(p)}{p^s} + g_3(s) = h \sum_q \frac{1}{q^s} + g_3(s),$$

$$\log f(s) = \frac{2}{h} \sum_{\varkappa=1}^{h} \chi_\varkappa(l) \log L_\varkappa(s) + g_4(s).$$

Es sei nun allgemeiner

$$a_n = 2^{\nu(n)}$$

oder

$$a_n = 0,$$

je nachdem alle $\nu(n)$ Primfaktoren der Zahl n einer von λ gegebenen Progressionen $am + b_1, \ldots, am + b_\lambda$ (wo $(a, b_1) = 1, \ldots$ ist) angehören oder nicht, d. h. es sei

$$a_n = 2^{\nu(n)}\Theta(n)$$

im Sinne des zweiten Teils der Einleitung. Dann ist die für $\sigma > 1$ durch

$$\psi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \equiv b_1, \ldots, b_\lambda} \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \ldots\right)$$

definierte analytische Funktion

$$= f_1(s) f_2(s) \ldots f_\lambda(s),$$

wo $f_1(s), \ldots, f_\lambda(s)$ die den einzelnen $b\ (= b_1, \ldots, b_\lambda)$ entsprechenden obigen $f(s)$ sind. Die für $s > 1$ reell definierte und für $\sigma > 1$ eindeutige Funktion

$$\omega(s) = \log \psi(s)$$

hängt also, zunächst für $\sigma > 1$, mit den $\log L_\kappa(s)$ durch die Gleichung zusammen:

$$\omega(s) = \frac{2}{h} \sum_{\alpha=1}^{\lambda} \sum_{\kappa=1}^{h} \chi_\kappa(l_\alpha) \log L_\kappa(s) + g_5(s). \tag{8}$$

Auf der rechten Seite von (8) hat $\log L_1(s)$ den Koeffizienten $\frac{2\lambda}{h}$; (8) schreibt sich also kurz folgendermassen:

$$\omega(s) = \sum_{\kappa=1}^{h} e_\kappa \log L_\kappa(s) + g_5(s),$$

wo

$$e_1 = \frac{2\lambda}{h}$$

ist und die Werte $e_2, \ldots, e_h$ nicht in Betracht kommen. Nach dem Satz am Ende des § 1 ist also in dem dort angegebenen einfach zusammenhängenden Gebiete (inkl. beider Ufer des Schnittes bis zum Punkt $s = 1$ ausschliesslich) $\omega(s)$ regulär und es ist für $|t| \geqq 3$, $\sigma \geqq 1 - \frac{1}{\log^c |t|}$

$$|\omega(s)| < k \log \log |t|.$$

Folglich ist

$$\psi(s) = e^{\omega(s)}$$

in jenem Gebiete regulär und für $|t| \geqq 3$, $\sigma \geqq 1 - \frac{1}{\log^c |t|}$ ist

$$|\psi(s)| = e^{\Re\omega(s)} \leqq e^{|\omega(s)|} < \log^k |t|. \tag{9}$$

In der Umgebung des Punktes $s = 1$ ist $\left(\text{mit einem Radius} > \frac{1}{\log^c 3}\right)$

$$\omega(s) = \frac{2\lambda}{h} \log \frac{1}{s-1} + \delta_0 + \delta_1 (s-1) + \ldots.,$$

$$\psi(s) = \frac{1}{(s-1)^{\frac{2\lambda}{h}}} (\gamma_0 + \gamma_1 (s-1) + \ldots.),$$

wo $\gamma_0 \neq 0$ ist. $\psi(s)$ unterscheidet sich an beiden Ufern des Schnittes um den Faktor $e^{\frac{4\pi\lambda i}{h}}$ (der natürlich für $\lambda = \frac{h}{2}$ und $\lambda = h$ den Wert 1 hat, so dass dann $s = 1$ ein Pol von $\psi(s)$ ist).

Ich bemerke ausdrücklich, dass ich nötig hatte, die obigen Sätze über $\Im \log L_\kappa(s)$ zu entwickeln, da die e_κ nicht notwendig reell sind, also bei

$$\Re\,\omega(s) = \sum_{\kappa=1}^{h} \Re\,(e_\kappa \log L_\kappa(s)) + \Re\, g_5(s)$$

die imaginären Teile der $\log L_\kappa(s)$ auftreten.

§ 3.

Nach der bekannten Integralformel

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{y^s}{s^2}\, ds \begin{cases} = \log y \text{ für } y \geqq 1, \\ = 0 \text{ für } 0 < y \leqq 1 \end{cases}$$

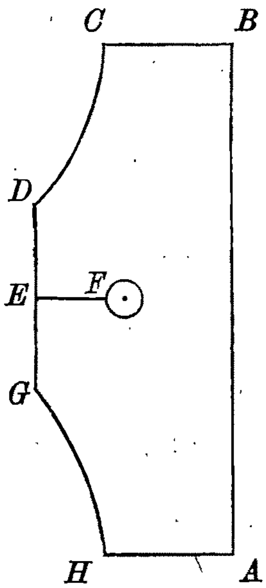
(bei geradem Integrationsweg) ist für $x > 0$

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2}\psi(s)ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \int_{2-\infty i}^{2+\infty i} \frac{\left(\frac{x}{n}\right)^s}{s^2} ds = \sum_{n=1}^{x} a_n \log\frac{x}{n},$$

also

$$\sum_{n=1}^{x} a_n \log\frac{x}{n} = \frac{1}{2\pi i}\int_{2-x^2 i}^{2+x^2 i} \frac{x^s}{s^2}\psi(s)\,ds + O\int_{x^2}^{\infty} \frac{x^2}{t^2}\,dt$$

$$= \frac{1}{2\pi i}\int_{2-x^2 i}^{2+x^2 i} \frac{x^s}{s^2}\psi(s)\,ds + O(1). \tag{10}$$



Auf den Integranden $\frac{x^s}{s^2}\psi(s)$ werde nun für $x > \sqrt{3}$ der Cauchy'sche Integralsatz bei folgendem in verkürzten Dimensionen gezeichneten Integrationsweg angewendet: Von $A = 2 - x^2 i$ geradlinig bis $B = 2 + x^2 i$; geradlinig bis $C = 1 - \frac{1}{\log^c (x^2)} + x^2 i$; auf der Kurve $s = 1 - \frac{1}{\log^c t} + ti$ bis $D = 1 - \frac{1}{\log^c 3} + 3i$; geradlinig bis $E = 1 - \frac{1}{\log^c 3}$; geradlinig am oberen Ufer des Schnittes bis $F = 1 - \varepsilon$, wo $1 - \varepsilon$ irgendwo zwischen $1 - \frac{1}{\log^c 3}$ und 1 gewählt sei; auf dem Kreise mit dem Radius ε um 1 von F im negativen Sinn bis F am unteren Ufer des Schnittes; auf diesem Ufer zurück bis E; geradlinig bis $G = 1 - \frac{1}{\log^c 3} - 3i$; auf der Kurve $s = 1 - \frac{1}{\log^c (-t)} + ti$ bis $H = 1 - \frac{1}{\log^c (x^2)} - x^2 i$; geradlinig bis A.

Nach dem Cauchy'schen Satz ist, da der Integrand auf dem Wege und in dem umlaufenen Gebiet regulär ist, das in (10) auftretende Integral

$$\int_A^B = -\int_B^C - \int_C^D - \int_D^E - \int_E^F - \int_F^F - \int_F^E - \int_E^G - \int_G^H - \int_H^A. \tag{11}$$

Nach (9) ist

$$\begin{aligned}
\left|\int_B^C\right| &= \left|\int_H^A\right| = O\left(\frac{\log^k x}{x^2}\right),\\
\left|\int_C^D\right| &= \left|\int_G^H\right| = O\int^{x^2} \frac{x^{1-\frac{1}{\log^c t}}}{t^2} \log^k t\, dt\\
&= O\left(x \log^k x \int_3^{x^2} \frac{x^{-\frac{1}{\log^c t}}}{t^2}\, dt\right)\\
&= O\left(x \log^k x \int_3^{e^{2c\sqrt{\log x}}} \frac{x^{-\frac{1}{\log^c t}}}{t^2}\, dt\right) + O\left(x \log^k x \int_{e^{2c\sqrt{\log x}}}^{x^2} \frac{x^{-\frac{1}{\log^c t}}}{t^2}\, dt\right)\\
&= O\left(x \log^k x\; x^{-\frac{1}{(2c\sqrt{\log x})^c}} \int_3^{e^{2c\sqrt{\log x}}} \frac{dt}{t^2}\right) + O\left(x \log^k x \int_{e^{2c\sqrt{\log x}}}^{x^2} \frac{dt}{t^2}\right)\\
&= O\left(x \log^k x\, e^{\sqrt{\log x}}\right) + O\left(x \log^k x\, e^{-2c\sqrt{\log x}}\right)\\
&= O\left(x e^{-2c\sqrt{\log x}}\right).
\end{aligned} \tag{12}$$

Ferner ist

$$\left|\int_D^E\right| = \left|\int_E^G\right| = O\left(x^{1-\frac{1}{\log^c 3}}\right). \tag{13}$$

Aus (10), (11), (12), (13) folgt

$$2\pi i \sum_{n=1}^{x} a_n \log\frac{x}{n} = O\left(xe^{-3\gamma\sqrt{\log x}}\right) - \int_{EFFE} \frac{x^s}{s^2}\psi(s)\,ds$$
$$= O\left(xe^{-3\gamma\sqrt{\log x}}\right) - J, \tag{14}$$

wo der Weg *EFFE* beim Integral J im negativen Sinne zu durchlaufen ist.

1) Es sei $\lambda < \frac{h}{2}$. Dann hat der Integrand für $s = 1$ eine algebraische Unendlichkeitsstelle der Ordnung $\frac{2\lambda}{h} < 1$; es darf also der Kreis in den Punkt $s = 1$ zusammengezogen werden, und das Integral ist einfach, $\frac{1}{\log^c 3} = \vartheta$ gesetzt,

$$J = \int_{1-\vartheta}^{1} \frac{x^s}{s^2}\psi(s)ds + \int_{1}^{1-\vartheta} \frac{x^s}{s^2}\psi(s)ds,$$

wo zuerst oben, dann unten integriert wird. Nun ist für $|s-1| < r$, wo $r > \vartheta$ ist,

$$\frac{\psi(s)}{s^2} = \frac{1}{s^2}\frac{1}{(s-1)^{\frac{2\lambda}{h}}}\left(\gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \ldots\right)$$
$$= \frac{1}{(s-1)^{\frac{2\lambda}{h}}}\left(\gamma_0 + \beta_1(s-1) + \beta_2(s-1)^2 + \ldots\right),$$

also, da für $1 - \vartheta \leqq s \leqq 1$

$$|\beta_1(s-1) + \beta_2(s-1)^2 + \ldots| \leqq B|s-1|$$

ist, wenn $(s-1)^{\frac{2\lambda}{h}}$ den Wert am oberen Rande bezeichnet,

$$J = \gamma_0\left(1 - e^{2\pi i\frac{2\lambda}{h}}\right)\int_{1-\vartheta}^{1} \frac{x^s}{(s-1)^{\frac{2\lambda}{h}}}ds + O\int_{1-\vartheta}^{1} \frac{x^s}{(1-s)^{\frac{2\lambda}{h}-1}}ds,$$

folglich,[12] wenn $\frac{2\lambda}{h} = \eta$ gesetzt wird $(0 < \eta < 1)$ und $(1-s)^\eta$ den reellen Wert bezeichnet,

$$J = \gamma \int_{1-\vartheta}^{1} \frac{x^s}{(1-s)^\eta} ds + O \int_0^1 \frac{x^s}{(1-s)^{\eta-1}} ds$$

$$= \gamma \int_0^1 \frac{x^s}{(1-s)^\eta} ds + O(x^{1-\vartheta}) + O \int_0^1 x^s (1-s)^{1-\eta} ds$$

$$= \gamma \int_0^1 \frac{x^{1-u}}{u^\eta} du + O(x^{1-\vartheta}) + O \int_0^1 x^{1-u} u^{1-\eta} du$$

$$= \gamma x \int_0^1 e^{-u \log x} u^{-\eta} du + O(x^{1-\vartheta}) + O\left(x \int_0^1 e^{-u \log x} u^{1-\eta} du \right)$$

$$= \frac{\gamma x}{(\log x)^{1-\eta}} \int_0^{\log x} e^{-v} v^{-\eta} dv + O(x^{1-\vartheta}) + O\left(\frac{x}{(\log x)^{2-\eta}} \int_0^{\log x} e^{-v} v^{1-\eta} dv \right),$$

$$\lim_{x=\infty} \frac{J}{\frac{x}{(\log x)^{1-\eta}}} = \gamma \int_0^\infty e^{-v} v^{-\eta} dv = \gamma \Gamma(1-\eta) \neq 0,$$

also nach (14)

$$\lim_{x=\infty} \frac{(\log x)^{1-\eta}}{x} \sum_{n=1}^{x} a_n \log \frac{x}{n} = -\frac{\gamma \Gamma(1-\eta)}{2\pi i}$$

$$= j. \tag{15}$$

Da $\gamma \neq 0$, also $j \neq 0$ ist, ist natürlich j reell und > 0; die genauere Diskussion von Realität und Vorzeichen, welche dies verifizieren muss, ist unnötig.

2) Es sei $\eta = 1$, d. h. $\lambda = \frac{h}{2}$. Dann hat $\psi(s)$ im Punkt $s = 1$ einen Pol erster Ordnung. An Stelle des Integrals über den Weg *EFFE* tritt also einfach das mit $-2\pi i x$ multiplizierte Residuum von $\psi(s)$ für $s = 1$, so dass auch hier (15) bewiesen ist.

3) Es sei $\frac{h}{2} < \lambda < h$, d. h. $1 < \eta < 2$. Dann wird sich auch das vorläufige Resultat (15) mit $j > 0$ ergeben; nur muss man etwas vorsichtiger rechnen, da nicht in $s = 1$ hineinintegriert werden darf. Es ist in der Umgebung von $s = 1$

$$\frac{\psi(s)}{s^2} = \frac{1}{(s-1)^\eta} (\gamma_0 + \beta_1(s-1) + \beta_2(s-1)^2 + \ldots),$$

---

[12] γ ist nicht Null, da $\gamma_0 \neq 0$ war.

also, da für $1-\vartheta \leq s \leq 1$

$$|\beta_1(s-1)+\beta_2(s-1)^2+\ldots\ldots| \leq B\,|s-1|$$

ist und

$$\frac{x^s}{(s-1)^{\eta-1}}$$

in $s=1$ hineinintegriert werden darf,

$$\int_{EFFE} x^s \frac{1}{(s-1)^\eta}(\beta_1(s-1)+\beta_2(s-1)^2+\ldots\ldots)\,ds = O\int_0^1 \frac{x^s}{(1-s)^{\eta-1}}\,ds$$
$$= O\left(\frac{x}{(\log x)^{2-\eta}}\int_0^{\log x} e^{-v}\,v^{1-\eta}\,dv\right) = O\left(\frac{x}{(\log x)^{2-\eta}}\right).$$

Die Behauptung (15) ist also bewiesen, wenn

$$\lim_{x=\infty} \frac{(\log x)^{1-\eta}}{x}\int_{EFFE}\frac{x^s}{(s-1)^\eta}\,ds \neq 0 \tag{16}$$

gezeigt werden kann.

Nun ergibt sich durch unbestimmte partielle Integration

$$\int \frac{x^s}{(s-1)^\eta}\,ds = -\frac{1}{\eta-1}\,\frac{x^s}{(s-1)^{\eta-1}} + \frac{1}{\eta-1}\log x\int\frac{x^s}{(s-1)^{\eta-1}}\,ds; \tag{17}$$

das erste Glied der rechten Seite von (17) liefert zu

$$\int_{EFFE}\frac{x^s}{(s-1)^\eta}\,ds$$

den Beitrag $O(x^{1-\vartheta})$; der Quotient des Integrals im zweiten Glied durch $\frac{x}{(\log x)^{2-\eta}}$ hat wegen $0<\eta-1<1$ nach dem Falle 1) einen Limes $\neq 0$. Damit ist (16), also (15) für $1<\eta<2$ bewiesen.

4) Falls $\eta=2$, d. h. $\lambda=h$ ist,[13] hat $\psi(s)$ im Punkte $s=1$ einen Pol zweiter Ordnung. Es ist alsdann

$$\frac{\psi(s)}{s^2} = \frac{\gamma_0}{(s-1)^2}+\frac{\beta_1}{s-1}+\beta_2+\ldots\ldots \quad (\gamma_0 \neq 0),$$

$$\frac{x^s\psi(s)}{s^2} = (x+(s-1)x\log x+\ldots\ldots)\frac{\psi(s)}{s^2} = \frac{\gamma_0 x}{(s-1)^2}+\frac{\beta_1 x+\gamma_0 x\log x}{s-1}+\ldots\ldots,$$

also

$$\sum_{n=1}^{x} a_n \log\frac{x}{n} = \gamma_0 x\log x + \beta_1 x + O\left(x e^{-\alpha\gamma\sqrt{\log x}}\right),$$

womit gleichfalls (15) bewiesen ist.

---

[13] Vergl. Anm. 5.

In jedem Fall liefert also dieser Paragraph das Ergebnis

$$\lim_{x=\infty} \frac{(\log x)^{1-\eta}}{x} \sum_{n=1}^{x} a_n \log \frac{x}{n} = j, \tag{15}$$

wo $j > 0$ ist.

§ 4.

Es sei $\delta > 0$ gegeben. Dann ist[14] nach (15)

$$\lim_{x=\infty} \frac{(\log x)^{1-\eta}}{x} \sum_{n=1}^{x+\delta x} a_n \log \frac{x+\delta x}{n} = j(1+\delta),$$

also durch Subtraktion

$$\lim_{x=\infty} \frac{(\log x)^{1-\eta}}{x} \left(\log(1+\delta) \sum_{n=1}^{x} a_n + \sum_{n=x+1}^{x+\delta x} a_n \log \frac{x+\delta x}{n}\right) = j\delta,$$

folglich für alle $x \geqq \xi_1 = \xi_1(\delta)$

$$\frac{j\delta}{1+\delta} \frac{x}{(\log x)^{1-\eta}} < \log(1+\delta) \sum_{n=1}^{x} a_n + \sum_{n=x+1}^{x+\delta x} a_n \log \frac{x+\delta x}{n} < j\delta(1+\delta) \frac{x}{(\log x)^{1-\eta}}. \tag{18}$$

Wenn

$$\sum_{n=1}^{x} a_n = A(x)$$

gesetzt wird, so ist

$$\log(1+\delta) A(x) \leqq \log(1+\delta) \sum_{n=1}^{x} a_n + \sum_{n=x+1}^{x+\delta x} a_n \log \frac{x+\delta x}{n} \leqq \log(1+\delta) A(x+\delta x). \tag{19}$$

Aus (18) und (19) ergibt sich für $x \geqq \xi_1(\delta)$

$$\log(1+\delta) A(x) < j\delta(1+\delta) \frac{x}{(\log x)^{1-\eta}} \tag{20}$$

und

$$\frac{j\delta}{1+\delta} \frac{x}{(\log x)^{1-\eta}} < \log(1+\delta) A(x+\delta x), \tag{21}$$

d. h., wenn in (21) $\frac{x}{1+\delta}$ statt x geschrieben wird, für $x \geqq (1+\delta)\xi_1 = \xi_2(\delta)$

$$A(x) > \frac{j\delta}{(1+\delta)\log(1+\delta)} \frac{\frac{x}{1+\delta}}{\left(\log \frac{x}{1+\delta}\right)^{1-\eta}} > \frac{j\delta}{(1+\delta)^2 \log(1+\delta)} \frac{x}{(\log x)^{1-\eta}}. \tag{22}$$

---

[14] Wegen $\lim\limits_{x=\infty} \frac{\log(x+\delta x)}{\log x} = 1$.

Nach (20) ist für $x \geqq \xi_1$

$$A(x) < \frac{j\delta(1+\delta)}{\log(1+\delta)} \frac{x}{(\log x)^{1-\eta}}. \tag{23}$$

Aus (22) folgt

$$\liminf_{x=\infty} \frac{(\log x)^{1-\eta}}{x} A(x) \geqq j \frac{\delta}{(1+\delta)^2 \log(1+\delta)} \tag{24}$$

für alle $\delta > 0$, d. h., da die rechte Seite von (24) für $\delta = 0$ den Limes j hat,

$$\liminf_{x=\infty} \frac{(\log x)^{1-\eta}}{x} A(x) \geqq j; \tag{25}$$

aus (23) folgt

$$\limsup_{x=\infty} \frac{(\log x)^{1-\eta}}{x} A(x) \leqq j \frac{\delta(1+\delta)}{\log(1+\delta)},$$

also

$$\limsup_{x=\infty} \frac{(\log x)^{1-\eta}}{x} A(x) \leqq j. \tag{26}$$

(25) und (26) ergeben vereinigt

$$\lim_{x=\infty} \frac{A(x)}{\frac{x}{(\log x)^{1-\eta}}} = a > 0.$$

Dies festzustellen war der Zweck der vorliegenden Abhandlung.

SCHLUSS.

Gleichzeitig erledigt sich hierdurch ein anderes Problem, von dem ich einen mit etwas elementareren Mitteln angreifbaren Spezialfall[15] an anderem Orte behandelt habe.[16] $B(x)$ sei die Anzahl der ganzen positiven Zahlen bis x, deren sämtliche Primfaktoren einer der λ Progressionen $am + b_1, \ldots, am + b_\lambda$ angehören.[17] Dann ist

$$B(n) - B(n-1) = \Theta(n)$$

[15] Es kamen dort nur reelle Charaktere in Betracht, so dass der Kunstgriff mit $\mathfrak{I} \log L_\kappa(s)$ nicht erforderlich war.

[16] "Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate" [*Archiv der Mathematik und Physik*, Ser. III, Bd. XIII (1908), S. 305–312].

[17] Wenn andere Primfaktoren unter der Beschränkung zugelassen werden, dass endlich viele derselben in beliebiger Potenz, die übrigen nur in gerader (oder mindestens in zweiter) Potenz auftreten — wie bei dem a. a. O. behandelten Problem — ändert sich natürlich nichts am Ergebnis. Denn es tritt einfach bei $\Psi(s)$ der Faktor

$$\prod_{p|P}\left(1+\frac{1}{p^s}+\ldots\right)\prod_p{}'\left(1+\frac{1}{p^{2s}}+\ldots\right) = e^{g_3(s)}$$

hinzu.

und für $\sigma > 1$

$$\Psi(s) = \sum_{n=1}^{\infty} \frac{\Theta(n)}{n^s} = \prod_{p \equiv b_1, \ldots, b_\lambda} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= e^{\sum\limits_{p \equiv b_1, \ldots, b_\lambda} \frac{1}{p^s} + g_6(s)}$$

$$= e^{\frac{1}{h} \sum\limits_{a=1}^{\lambda} \sum\limits_{\kappa=1}^{h} \chi_\kappa(l_a) \log L_\kappa(s) + g_7(s)}$$

$$= e^{\sum\limits_{\kappa=1}^{h} E_\kappa \log L_\kappa(s) + g_7(s)},$$

wo

$$E_1 = \frac{\lambda}{h}$$

ist. Also gelten die obigen Entwickelungen mit dem einzigen Unterschiede, dass $\eta = \frac{\lambda}{h}$ an Stelle von $\eta = \frac{2\lambda}{h}$ zu setzen ist, und es ergibt sich

$$\lim_{x=\infty} \frac{B(x)}{\dfrac{x}{(\log x)^{1-\frac{\lambda}{h}}}} > 0.$$

BERLIN, *den* 21. *Juni* 1908.

Rational Reduction of a Pair of Binary Quadratic Forms; their Modular Invariants.

By Leonard Eugene Dickson.

1. The primary object of the present paper is a study of the invariants of a pair of binary quadratic forms under modular transformation. Incidentally, the invariants of a single form are given a more satisfactory expression than hitherto employed (§ 7).

It is shown that the knowledge of a complete set of canonical types of pairs of forms is of great service in the discovery and proof of relations between certain of the modular invariants and in establishing the independence of other invariants (§§ 23, 25). For these reasons and for the purpose of giving interpretations to the modular invariants, we begin the investigation with a discussion of the necessary and sufficient conditions for the equivalence of two pairs of quadratic forms.

Within the field C of all complex numbers, Weierstrass's elementary divisors enable one to state necessary and sufficient conditions for the equivalence of two pairs of quadratic forms; but for a smaller field contained in C, or for any finite field, these conditions are not sufficient, since the formulae of transformation involves irrationalities. Before stating the additional necessary conditions, we express the above conditions in the following equivalent form. Let θ denote the quadratic simultaneous invariant and j the Jacobian of q_1, q_2; Θ and J those for Q_1, Q_2. If $j \not\equiv 0$, then must

$$| Q_1 | : | q_1 | = | Q_2 | : | q_2 | \doteq \Theta : \theta .$$

But if $j \equiv 0$, so that $q_2 \equiv m q_1$, then must $J \equiv 0$, so that $Q_2 \equiv M Q_1$; and furthermore, m and M must be equal. For a field other than C, the above equal ratios, as well as certain other specified functions of the coefficients, must be squares in the field; further, the leading coefficient of Q_1 must be representable by the form q_1. The latter condition, requiring the solvability of a

diophantine equation, is considerably weaker than the requirement that an indicated square root shall be rational.

For finite fields, the criteria become simpler and are expressed entirely in terms of the invariants of the two forms. When the modulus p exceeds 2, the criteria are that the algebraic invariants $|q_1|$, $|q_2|$, θ of one pair of forms shall equal the products of those of the other pair by the same square, and that four modular absolute invariants have the same values for the two pairs (§ 13). For $p=2$, we again employ three relative invariants, the resultant and the coefficients of xy (which take the place of the determinants), and three absolute invariants (§§ 22, 32).

REDUCTION IN A FIELD F NOT HAVING MODULUS 2, §§ 2–13.

2. Consider two quadratic forms with coefficients in F,

$$q_1 = a_0 x^2 + 2a_1 xy + a_2 y^2, \quad q_2 = b_0 x^2 + 2b_1 xy + b_2 y^2, \tag{1}$$

having the determinants and simultaneous invariant

$$a = a_0 a_2 - a_1^2, \quad b = b_0 b_2 - b_1^2, \quad \theta = a_0 b_2 - 2a_1 b_1 + a_2 b_0. \tag{2}$$

Consider a second pair of forms Q_1, Q_2, with coefficients $A_0, \ldots, B_2$ in F and invariants A, B, Θ. If there exists in F a linear tranformation of determinant Δ which replaces q_1 by Q_1, and q_2 by Q_2, the product of the determinant

$$|\lambda q_1 + \mu q_2| = a\lambda^2 + \theta\lambda\mu + b\mu^2 \tag{3}$$

by Δ^2 equals the determinant

$$|\lambda Q_1 + \mu Q_2| = A\lambda^2 + \Theta\lambda\mu + B\mu^2. \tag{4}$$

Hence a necessary condition for the equivalence of the two pairs is*

$$A : a = \Theta : \theta = B : b = \text{square in } \mathsf{F}. \tag{5}$$

3. First, let q_1 and Q_1 be irreducible in F, viz., let $-a$ and $-A$ be not-squares. In particular, $a_0 \neq 0$, $A_0 \neq 0$. For $x = X - a_1 Y$, $y = a_0 Y$,

$$q_1 = a_0(X^2 + aY^2), \quad q_2 = b_0 X^2 + 2cXY + dY^2, \tag{6}$$

$$c = a_0 b_1 - a_1 b_0, \quad d = b_0 a_1^2 - 2b_1 a_0 a_1 + b_2 a_0^2. \tag{7}$$

By (5), $a = t^2 A$, t an element in F. For $x = \xi - A_1 t\eta$, $y = A_0 t\eta$,

$$Q_1 = A_0(\xi^2 + a\eta^2), \quad Q_2 = B_0\xi^2 + 2C\xi\eta + D\eta^2, \tag{8}$$

$$C = t(A_0 B_1 - A_1 B_0), \quad D = t^2(B_0 A_1^2 - 2B_1 A_0 A_1 + B_2 A_0^2). \tag{9}$$

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* In the sense $A = \Delta^2 a$, etc., so that $a = 0$ implies $A = 0$, etc.

Then $X=\alpha\xi+\beta\eta$, $Y=\gamma\xi+\delta\eta$ replaces q_1 by Q_1, if, and only if,

$$a_0(\alpha^2+a\gamma^2)=A_0,\quad \alpha\beta+a\gamma\delta=0,\quad a_0(\beta^2+a\delta^2)=A_0 a.$$

Eliminating β from the last two and applying the first, we get $\delta^2=\alpha^2$. In every case, we have $\delta=\pm\alpha$, $\beta=\mp a\gamma$. The only further condition is

$$a_0(\alpha^2+a\gamma^2)=A_0, \tag{10}$$

which states that the form q_1 must be capable of representing A_0. We assume that this necessary condition is satisfied and let α, γ be a particular set of solutions in F of (10). Then

$$X=\alpha\xi-a\gamma\eta,\quad Y=\gamma\xi+\alpha\eta$$

transforms the pair of forms (6) into

$$q_1=A_0(\xi^2+a\eta^2),\quad q_2=e\xi^2+2f\xi\eta+g\eta^2, \tag{11}$$

$$\left.\begin{array}{c} e=b_0\alpha^2+2c\alpha\gamma+d\gamma^2,\quad f=c\alpha^2+(d-b_0a)\alpha\gamma-ca\gamma^2,\\ g=d\alpha^2-2ca\alpha\gamma+b_0a^2\gamma^2. \end{array}\right\} \tag{12}$$

Now $q_1=Q_1$. Hence any transformation T replacing q_1, q_2 by Q_1, Q_2 must be an automorph of q_1. By the above discussion, T must be of the type

$$\xi=r\xi'\mp as\eta',\quad \eta=s\xi'\pm r\eta',\quad r^2+as^2=1. \tag{13}$$

We proceed to express (13) in parametric form. For $s\neq 0$, we may set

$$r-1=\rho s\ (\rho\neq 0),\quad r+1=-as/\rho.$$

The resulting values of r, s are given by (14) for $\sigma=1$:

$$r=\frac{-\rho^2+a\sigma^2}{\rho^2+a\sigma^2},\quad s=\frac{-2\rho\sigma}{\rho^2+a\sigma^2}. \tag{14}$$

The sets $s=0$, $r=\pm 1$, are given by (14) for $\sigma=0$, or $\rho=0$. Hence the solutions of $r^2+as^2=1$ are given uniquely by (14) with ρ and σ not both zero, so that the denominators do not vanish. Now (13) replaces (11_2) by

$$q_2=E\xi'^2+2F\xi'\eta'+G\eta'^2,\quad E=er^2+2frs+gs^2; \tag{15}$$

the values of F and G not being required. In fact, since (13) has determinant ± 1, we have the absolute invariants

$$\Delta\equiv f^2-eg=F^2-EG,\quad I\equiv g+ea=G+Ea. \tag{16}$$

As a temporary abbreviation, set

$$k=g-ea,\quad l=\tfrac{1}{2}(E-e). \tag{17}$$

Then by (14) and (15),

$$l = \{ k\rho^2\sigma^2 + f\rho\sigma(\rho^2 - a\sigma^2) \} / (\rho^2 + a\sigma^2)^2. \tag{18}$$

Postponing the cases $\rho = 0$ and $\sigma = 0$, we may take $\rho = 1$, $\sigma \neq 0$. Then (18) is unaltered when σ is replaced by $-1/a\sigma$. Thus we set

$$\varepsilon = \sigma - 1/a\sigma. \tag{19}$$

Then (18) becomes the quadratic equation

$$l(a\varepsilon^2 + 4) = k/a - f\varepsilon. \tag{20}$$

But by (16) and (17),

$$F^2 - f^2 = E(I - Ea) - e(I - ea) = 4l[I - a(E + e)] = 4l(k - 4al). \tag{21}$$

Hence (20) gives

$$(al\varepsilon + \tfrac{1}{2}f)^2 = lk - 4al^2 + \tfrac{1}{4}f^2 = \tfrac{1}{4}F^2,$$

$$\varepsilon = \varepsilon_\pm = (f \pm F)/(-2al). \tag{22}$$

By (19) and (20),

$$(\sigma - \tfrac{1}{2}\varepsilon)^2 = (a\varepsilon^2 + 4)/4a = (k/a - f\varepsilon)/(4al). \tag{23}$$

Inserting the value (22) and eliminating k by (21), we get

$$(\sigma - \tfrac{1}{2}\varepsilon)^2 = S_\pm/(16a^2l^2), \quad S_\pm \equiv (F \pm f)^2 + a(E - e)^2. \tag{24}$$

Hence one of the $S_\pm$ must be zero or a square in the field F. The same result holds if $\rho = 0$ or if $\sigma = 0$, since then $l = 0$, $E = e$, $G = g$, $F^2 = f^2$.

THEOREM. *The necessary and sufficient conditions that a pair of quadratic forms* (1), *of which the first is irreducible in the field* F, *shall be equivalent in* F *to a pair* Q_1 *and* Q_2 *are that relations* (5) *shall hold between their invariants, that* A_0 *shall be representable by the form* q_1, *and that one of the expressions*

$$(C \pm f)^2 + a(B_0 - e)^2 \tag{24'}$$

shall be a square in F, *where* C, f, e *are given by* (7), (9), (10), (12).

For a finite field, equation (10) is solvable,* so that A_0 is representable by q_1. Further, the condition on (24′) is satisfied if $-R$ is zero or a not-square, where

$$R = 4ab - \theta^2 \tag{25}$$

is the resultant of (1). Indeed, by (22), (21),

$$\left(\frac{k}{a} - f\varepsilon_+\right)\left(\frac{k}{a} - f\varepsilon_-\right) = \frac{k^2}{a^2} + \frac{kf^2}{a^2 l} + \frac{f^2(f^2 - F^2)}{4a^2l^2} = \frac{k^2 + 4af^2}{a^2}.$$

* *Linear Groups*, p. 46.

But by (16), (17), and $I = A_0\theta$, $\Delta = -A_0^2 b$,

$$k^2 + 4af^2 = (I - 2ae)^2 + 4a(\Delta + Ie - ae^2) = I^2 + 4a\Delta = -A_0^2 R.$$

Hence by (23), (24),

$$S_+ S_- = -16\, l^2 A_0^2 R. \tag{26}$$

In a finite field the product of two not-squares is a square; hence if $-R$ is zero* or a not-square one of the $S_\pm$ is zero or a square.

COROLLARY. *In a finite field a pair of quadratic forms* (1), *whose resultant R is zero or the negative of a not-square, and the first of which is irreducible, is equivalent to a second pair if and only if relations* (5) *hold between their invariants.*

4. The case in which $-R$ is a square $\neq 0$ in F, while q_1 is irreducible, may be treated advantageously by a well known method. Then (3) vanishes for two distinct values of λ/μ in F. Hence the family contains two distinct forms each a multiple of a perfect square. Thus by a linear transformation in F we may replace the pair (1) by a pair

$$q_1 = a_0(x^2 + ay^2), \quad q_2 = b_0(x^2 + by^2), \quad a_0 b_0 \neq 0, \quad a \neq b, \tag{27}$$

of resultant $-a_0^2 b_0^2 (a-b)^2$. The first form of an equivalent pair may be taken to be $A_0(X^2 + AY^2)$, where $A = d^2 a$. For $X = \xi$, $Y = \eta/d$,

$$Q_1 = A_0(\xi^2 + a\eta^2), \quad Q_2 = B_0(\xi^2 + B\eta^2), \quad A_0 B_0 \neq 0, \quad a \neq B. \tag{28}$$

As shown in § 3, the transformations of q_1 into Q_1 are

$$x = \alpha\xi \mp a\gamma\eta, \quad y = \gamma\xi \pm \alpha\eta, \quad a_0(\alpha^2 + a\gamma^2) = A_0. \tag{29}$$

This will transform q_2 into Q_2 if, and only if,

$$b_0(\alpha^2 + b\gamma^2) = B_0, \quad \alpha\gamma(a-b) = 0, \quad b_0(a^2\gamma^2 + b\alpha^2) = B_0 B.$$

Now $a \neq b$. According as $\gamma = 0$ or $\alpha = 0$, we have

$$B = b, \qquad A_0/a_0 = B_0/b_0 = \text{square in } \mathsf{F} \text{ (viz., } \alpha^2); \tag{30}$$

$$B = a^2/b, \quad A_0/a_0 a = B_0/b_0 b = \text{square in } \mathsf{F} \text{ (viz., } \gamma^2), \quad b \neq 0. \tag{31}$$

For the pair (27), the determinant (3) becomes

$$a_0^2 a\lambda^2 + a_0 b_0(a+b)\lambda\mu + b_0^2 b\mu^2$$

and vanishes for $\lambda/\mu = -b_0/a_0$, $-bb_0/aa_0$. For (4), the roots are $-B_0/A_0$, $-BB_0/aA_0$. The conditions for the identity of the two sets of roots are (30) or (31), apart from the requirement that the ratios be squares. The latter is

* Then $k^2 + 4af^2 = 0$, $k = f = 0$, so that the second form (11) is a multiple of the first. This is evident since the forms have a common root and the first is irreducible.

therefore a condition additional to those in the algebraic theory. For a finite field, it is shown in § 10 that the two pairs of forms are equivalent if their algebraic invariants satisfy (5), and if two modular invariants have equal values.

5. Finally, let q_1 be reducible in the field, the necessary and sufficient condition for which is $-a =$ square or zero. Then q_1 may be given one of the types $2xy$, a_0x^2. By (5), $-A =$ square or zero. After a preliminary transformation we may take Q_1 to be $2xy$ or A_0x^2. Then let q_2 and Q_2 have the coefficients b_i and B_i, respectively.

The automorphs of $2xy$ are $(kx, k^{-1}y)$, $(ky, k^{-1}x)$. Hence must

$$B_1 = b_1,\ B_0 = k^2 b_0,\ B_2 = k^{-2} b_2;\ \text{or}\ B_1 = b_1,\ B_0 = k^{-2} b_2,\ B_2 = k^2 b_0. \quad (32)$$

Necessary and sufficient conditions for equivalence are that $B_1 = b_1$, $B_0B_2 = b_0b_2$ (to which (5) now reduce), and that if $b_i \neq 0$ ($i = 0$ or 2) one of the ratios of B_0, B_2 to b_i shall be a square $\neq 0$ in the field; while if $b_0 = b_2 = 0$, then* $B_0 = B_2 = 0$. For a finite field the last conditions may be expressed by a modular invariant (§ 11).

6. For $q_1 = a_0x^2$, $Q_1 = A_0x^2$, a necessary condition for equivalence is $A_0 = t^2a_0$. The general transformation of the first pair into the second is then $(tx,\ rx + sy)$, where

$$B_0 = b_0t^2 + 2b_1rt + b_2r^2, \quad B_1 = b_1st + b_2rs, \quad B_2 = b_2s^2. \quad (33)$$

For $a_1 = a_2 = 0$, conditions (5) reduce to $A_0B_2 : a_0b_2 = B : b =$ square. These, with $A_0/a_0 =$ square, are sufficient if $b_2 \neq 0$ or if $b_2 = 0$, $b_1 \neq 0$ (whence $B_2 = 0$, $B_1 \neq 0$); since $A_0 = t^2a_0$ and (33) may then be satisfied by choice of t, s, r in the field. The condition $A_0/a_0 =$ square may be expressed in a finite field by the modular invariant Q_a (§ 12). If† $b_1 = b_2 = 0$, (5) give $B_1 = B_2 = 0$; further necessary conditions are $B_0 : b_0 = A_0 : a_0 =$ square (viz., t^2). For a finite field the latter conditions may be expressed by the modular invariants Q_a and K_1 (§ 12).

7. Every binary linear homogeneous transformations with coefficients in a given field can be generated by the three types

$$x = x' + ty', \quad y = y'; \quad (34)$$

$$x = y', \quad y = -x'; \quad (35)$$

$$x = x', \quad y = \lambda y'; \quad (36)$$

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* For $q_1 = Q_1 = 2xy$, $q_2 = 2b_1xy$, $Q_2 = B_0x^2 + 2b_1xy + B_2y^2$, the minors of $|\lambda q_1 + \mu q_2|$ have the factor $\lambda + \mu b_1$. Those of $|\lambda Q_1 + \mu Q_2|$ have the same factor if and only if $B_0 = B_2 = 0$. See § 1.

† Then $q_1 = a_0x^2$, $q_2 = b_0x^2$, and the minors of $|\lambda q_1 + \mu q_2|$ have the factor $\lambda a_0 + \mu b_0$. For a second pair of such forms, the factor is $\lambda A_0 + \mu B_0$. The relative invariance of this factor leads to the condition $B_0 : b_0 = A_0 : a_0$. See § 1.

where t and λ are arbitrary non-vanishing elements of the field. Under these transformations the forms (1) become q_1', q_2', with the coefficients

$$\left.\begin{array}{l} a_0'=a_0,\quad a_1'=a_1+t\,a_0,\quad a_2'=a_2+2\,t\,a_1+t^2 a_0, \\ b_0'=b_0,\quad b_1'=b_1+t\,b_0,\quad b_2'=b_2+2\,t\,b_1+t^2 b_0; \end{array}\right\} \tag{37}$$

$$a_0'=a_2,\ a_1'=-a_1,\ a_2'=a_0,\ b_0'=b_2,\ b_1'=-b_1,\ b_2'=b_0; \tag{38}$$

$$a_i'=\lambda^i a_i,\quad b_i'=\lambda^i b_i\qquad (i=0,1,2). \tag{39}$$

Let the field be the Galois field $GF[p^n]$ of order p^n, $p>2$. Set

$$\tau=\tfrac{1}{2}(p^n-1). \tag{40}$$

If C denotes a binomial coefficient, we have

$$C_i^{2\tau}\equiv(-1)^i,\quad (k-l)^{2\tau}\equiv\sum_{i=0}^{2\tau} k^i\, l^{2\tau-i}\qquad (\mathrm{mod}\ p). \tag{41}$$

For the invariant $a=a_0 a_2-a_1^2$ of q_1, we have

$$a^{2\tau}\equiv\sum_{i=0}^{\tau-1} a_0^i\, a_2^i\, a_1^{4\tau-2i}+\sum_{j=\tau}^{2\tau} a_0^j\, a_2^j\, a_1^{4\tau-2j}.$$

To the first sum we apply

$$a_1^{2\tau+r}=a_1^r\qquad (r>0). \tag{42}$$

In the last sum we set $j=\tau+i$. Hence

$$a^{2\tau}-1=(a_0^\tau a_2^\tau+1)\,\sigma,\quad \sigma\equiv\sum_{i=0}^{\tau} a_0^i\, a_2^i\, a_1^{2\tau-2i}-1. \tag{43}$$

We may now show that q_1 has the absolute invariant

$$Q=(a_0^\tau+a_2^\tau)\,\sigma. \tag{44}$$

Obviously Q is absolutely invariant under (38) and (39). It remains to establish its invariance under (37). Let the latter give to a_2^τ and σ the increments δ and σ_1. Then the increments to $a^{2\tau}-1$ and Q are

$$(a_0^\tau a_2^\tau+1)\,\sigma_1+a_0^\tau\,\delta\,(\sigma+\sigma_1)=0,\qquad (a_0^\tau+a_2^\tau)\,\sigma_1+\delta\,(\sigma+\sigma_1).$$

If $a_0\neq 0$, we multiply the former by a_0^τ and obtain the latter, since $a_0^{2\tau}=1$. If $a_0=0$, then $\sigma=a_1^{2\tau}-1$, $a_1\sigma=0$, so that Q is unaltered by $a_2'=a_2+2\,t\,a_1$.

Since $a_i\,(a_i^{2\tau}-1)=0$ in the field, we have by (43), (44),

$$(a^{2\tau}-1)^2-Q^2=(a_0^{2\tau}-1)\,(a_2^{2\tau}-1)\,\sigma^2=(a_0^{2\tau}-1)\,(a_2^{2\tau}-1)\,(a_1^{2\tau}-1)^2.$$

But $(k^{2\tau}-1)^2=-(k^{2\tau}-1)$. Hence*

$$a^{2\tau}-1+Q^2=I=(a_0^{2\tau}-1)\,(a_1^{2\tau}-1)\,(a_2^{2\tau}-1). \tag{45}$$

* Concerning invariant I, see *Trans. Amer. Math. Soc.*, Vol. VIII (1907), p. 206.

Multiplying this by a and applying the obvious relation $aI=0$, we get $aQ^2=0$. Hence $aQ=0$.

A complete set* of independent invariants of q_1 is given by a and Q. Let ν be a fixed not-square. Then q_1 can be reduced by a linear transformation in the $GF[p^n]$, $p>2$, to one and but one of the forms

$$\begin{array}{llllll} q_1 = x^2-\nu y^2, & 2xy, & x^2, & \nu x^2, & \text{Identically zero;} \\ a=-\nu, & -1, & 0, & 0, & 0; \\ Q=0, & 0, & -1, & +1, & 0. \end{array}$$

Two forms are equivalent if and only if they have the same Q and a^τ.

8. If we replace each a_i by a_i+kb_i in the determinant a of q_1, we obtain $a+k\theta+k^2b$, where θ is the simultaneous invariant (2) of q_1 and q_2. From Q_a we obtain similarly new simultaneous invariants K_i:

$$Q_{a+kb} = Q_a + k^\tau Q_b + \sum_{i=1}^{2\tau} k^i K_i, \tag{46}$$

the exponents $>2\tau$ of k having been reduced by $k^{2\tau+\tau}=k^\tau$. We shall be able to apply the invariants K_i without obtaining their explicit expressions.

For the case $p^n=3$, we have

$$\left.\begin{array}{l} K_1 = a_0^2b_2+a_2^2b_0+a_1^2b_2+a_1^2b_0-a_0a_2b_2-a_0a_2b_0-a_0a_1b_1-a_1a_2b_1, \\ K_2 = b_0^2a_2+b_2^2a_0+b_1^2a_2+b_1^2a_0-b_0b_2a_2-b_0b_2a_0-b_0b_1a_1-b_1b_2a_1, \end{array}\right\} \tag{47}$$

K_1 and K_2 being interchanged when the a's and b's are interchanged.

9. We may now readily derive a complete set of non-equivalent canonical types of a pair of binary quadratic forms in the $GF[p^n]$, $p>2$, the various types being invariantly characterized. We begin with the case in which q_1 is irreducible in the field, while the resultant R, given by (25), is zero or the negative of a not-square. By § 3, we may take $q_1=x^2-\nu y^2$, ν being a fixed not-square; $q_2=mq_1$ if $R=0$; $q_2=ex^2+2fxy+gy^2$ if $-R$ is a not-square, where, for arbitrary elements b and θ for which $\theta^2+4\nu b$ is a not-square $(-R)$, e, f, g is a particular set of solutions of $eg-f^2=b$, $g-\nu e=\theta$. Thus e is a fixed element for which $\lambda\equiv e(\theta+\nu e)-b$ is a square, f is a fixed square root of λ, while $g=\theta+\nu e$. In either case, two such pairs of forms are equivalent only if they have equal values of the invariants b, θ (since the a's are equal).

10. Next, let q_1 be irreducible, and $-R$ be a square $\neq 0$. In (27), (28), we may set $a=-\nu$, $a_0=1$ or ν, $A_0=1$ or ν. Conditions (30) apply only

* For proof in special fields see *ibid.*, §§ 8, 13.

when A_0/a_0 is a square (whence $A_0 = a_0$) and are then trivial. Hence equivalence arises only when (31) can be satisfied. For $b = 0$, the canonical types are

$$q_1 = a_0 (x^2 - \nu y^2), \quad q_2 = b_0 x^2 \quad (a_0 = 1 \text{ or } \nu,\ b_0 \neq 0). \tag{48}$$

Consider (31) for $b \neq 0$, $a = -\nu$. If -1 is a not-square, A_0/a_0 must be a square, whence $A_0 = a_0$, $B_0 = -b_0 b/\nu$, $B = \nu^2/b$; the canonical types are

$$q_1 = a_0 (x^2 - \nu y^2), \quad q_2 = b_0 (x^2 + b y^2) \ (a_0 = 1 \text{ or } \nu,\ b_0 \neq 0,\ b \neq 0, -\nu), \tag{49}$$

only one of each pair (b_0, b), $(-b_0 b/\nu, \nu^2/b)$ being retained.*

If -1 is a square, (31) requires that A_0/a_0 be a not-square. Taking $a_0 = 1$, $A_0 = \nu$, we have $B_0 = -b_0 b$, $B = \nu^2/b$; for these values (27) and (28) are equivalent. Hence for -1 a square, the canonical types are

$$q_1 = x^2 - \nu y^2, \quad q_2 = b_0 (x^2 + b y^2) \quad (b_0 \neq 0,\ b \neq 0, -\nu), \tag{50}$$

and no two such pairs are equivalent. However, the pair (50) has the same determinants and the same value of θ as the similar pair with $B_0 = -b_0 b/\nu$, $B = \nu^2/b$, but not for any further pairs. Hence new invariants are required to distinguish two such pairs. Similar remarks apply to (48) and to (49).

To this end we determine the value of the absolute invariants $K_{2\tau}$ and† K_1, defined by (46), for the case $a_1 = b_1 = 0$. Then Q_{a+kb} becomes

$$Q'_{a+kb} = F_{02} + F_{20}, \quad F_{02} = (a_0 + k b_0)^{2\tau} (a_2 + k b_2)^{\tau} - (a_0 + k b_0)^{\tau}, \tag{51}$$

F_{20} being derived from F_{02} by interchanging a_0 with a_2 and b_0 with b_2. By (41),

$$(a_0 + k b_0)^{2\tau} \equiv \sum_{i=0}^{2\tau} (-1)^i k^i b_0^i a_0^{2\tau - i}.$$

The coefficient of $k^{2\tau}$ in F_{02} is therefore ‡

$$\sum_{j=0}^{\tau} [c_j^{\tau} b_2^j a_2^{\tau-j}] \, [(-1)^{2\tau-j} b_0^{2\tau-j} a_0^j].$$

Applying (02) to the subscripts, we obtain the required terms in F_{20}. Set $a_2 = a_0 a$, $b_2 = b_0 b$, as in (27). Then the terms free of a_1, b_1 in $K_{2\tau}$ are

$$K'_{2\tau} = a_0^{\tau} b_0^{2\tau} \sum_{j=0}^{\tau} (-1)^j c_j^{\tau} (a^{\tau-j} b^j + a^j b^{2\tau-j}).$$

---

* For example, if b is a not-square, we may restrict b_0 to the squares; if b is a square it may be restricted to the squares β for which the pairs $(\beta, \nu^2/\beta)$ yield all the squares $\neq 0, -\nu$,

† In §11 we employ K_τ. But for $a_1 = b_1 = 0$, $K_\tau = a_0^{2\tau} b_0^{\tau} [a^\tau + (-1)^\tau] (a - b)^\tau = 0$, since $(-a)^\tau + 1$ equals $\nu^\tau + 1 = 0$.

‡ There are no further terms in $k^{2\tau}$ obtained from $k^{4\tau} = k^{2\tau}$, etc.

In the first sum replace j by $\tau - j$, and hence $\tau - j$ by j. Thus, for $b_0 \neq 0$,

$$K'_{2\tau} = a_0^\tau [(-1)^\tau + b^\tau] [\sum_{j=0}^{\tau} (-1)^j c_j^\tau a^j b^{\tau-j}] = a_0^\tau [(-1)^\tau + b^\tau] (b - a)^\tau. \quad (52)$$

The sum of the coefficients of k and $k^{2\tau+1} \equiv k$ in F_{02} is

$$\tau a_0^{2\tau} a_2^{\tau-1} b_2 + 2\tau a_0^{2\tau-1} b_0 a_2^\tau - \tau a_0^{\tau-1} b_0 + \sum_{j=1}^{\tau} c_j^\tau b_2^j a_2^{\tau-j} (-1)^{1-j} b_0^{2\tau+1-j} a_0^{j-1}.$$

Replace a_2 by $a_0 a$, b_2 by $b_0 b$. Let $p^n > 3$, so that $\tau > 1$, $a_0^{3\tau-1} = a_0^{\tau-1}$. Then the terms free of a_1, b_1 in K_1 are

$$K'_1 = a_0^{\tau-1} b_0 \{\tau a^{2\tau} - \tau - a^{2\tau-1} b - a^\tau + \sum_{j=1}^{\tau} (-1)^{1-j} c_j^\tau (a^{\tau-j} b^j + a^{j-1} b^{2\tau+1-j})\}.$$

In the final terms of the sum replace j by $\tau + 1 - j$. There results

$$\sum_{j=1}^{\tau} (-1)^{\tau-j} c_{j-1}^\tau a^{\tau-j} b^{\tau+j}.$$

We shall employ K'_1 only for $a \neq 0$, $b^\tau = (-1)^{\tau+1}$. Then

$$K'_1 = a_0^{\tau-1} b_0 \{-a^{-1} b - a^\tau + \sum_{j=1}^{\tau} (-1)^{1-j} (c_j^\tau + c_{j-1}^\tau) a^{\tau-j} b^j\}.$$

Since $c_j^\tau + c_{j-1}^\tau = c_j^{\tau+1}$, we have

$$K'_1 = -a_0^{\tau-1} b_0 a^{-1} (a - b)^{\tau+1}, \text{ if } b^\tau = (-1)^{\tau+1}, \ \tau > 1. \quad (53)$$

For the forms (48), $K_{2\tau} = -(-1)^\tau a_0^\tau$, by (52), so that a_0^τ and hence also a_0 is absolutely invariant. Thus $|q_1| = -a_0^2 \nu$ is invariant. Then by (5), $\theta = -a_0 b_0 \nu$ and hence also b_0 is absolutely invariant. Thus $K_{2\tau}$, a and θ differentiate the forms (48).

For (49) we have -1 a not-square. We set $\nu = -1$. Then

$$|q_1| = a_0^2 = 1, \quad |q_2| = b_0^2 b, \quad \theta = a_0 b_0 (b + 1).$$

For such pairs of forms, the above are absolute invariants. From

$$A_0 = \pm a_0, \quad B_0^2 B = b_0^2 b, \quad A_0 B_0 (B + 1) = a_0 b_0 (b + 1),$$

we obtain, by eliminating B and A_0,

$$(B_0 \mp b_0)(b_0 b / B_0 \mp 1) = 0.$$

We need only examine the sets $(\pm b_0, b)$, since the other sets $(\pm b_0 b, 1/b)$ are not retained in the types (49). By (52), (53),

$$K_{2\tau} = a_0^\tau (b^\tau - 1)(b + \nu)^\tau; \quad K_1 = a_0^{\tau-1} b_0 \nu^{-1} (b + \nu)^{\tau+1} \text{ if } b^\tau = 1,$$

since τ is odd and > 1 for b a square ($b \neq 0$, $-\nu$ implies $b = \nu$ when $p^n = 3$).

When a_0 and b_0 are changed in sign, so also are $K_{2\tau}$ and K_1, and at least one is not zero for each $b \neq 0$. Hence the invariants differentiate the forms (49).

Finally, for (50) we have -1 a square, τ even. Then

$$K_{2\tau} = (b^\tau + 1)(b + \nu)^\tau; \quad K_1 = -\nu^{-1} b_0 (b + \nu)^{\tau+1} \text{ if } b^\tau = -1.$$

Each is changed in sign when b_0 is replaced by $-b_0 b/\nu$, and b by ν^2/b. Hence the invariants a, b, θ, $K_{2\tau}$, K_1 differentiate the forms (50).

11. To differentiate pairs of forms of which q_1 is $2xy$, we employ K_τ, defined by (46), for $a_0 = a_2 = 0$, $a_1 = 1$. Then Q_{a+kb} becomes

$$k^\tau (b_0^\tau + b_2^\tau) \{-1 + \sum_{i=0}^{\tau} k^{2i} b_0^i b_2^i (1 + k b_1)^{2\tau - 2i}\}.$$

Since the constant terms within the brackets cancel, terms in $k^\tau = k^{3\tau}$ are obtained only by employing the term of highest degree in the final binomial. Hence the coefficient of $k^{3\tau}$ is

$$(b_0^\tau + b_2^\tau) \sum_{i=0}^{\tau} b_0^i b_2^i b_1^{2\tau - 2i}.$$

This must equal $Q_b + K_\tau''$, where K_τ'' is the value of K_τ for $a_0 = a_2 = 0$, $a_1 = 1$. Hence, by (44),

$$K_\tau'' = b_0^\tau + b_2^\tau.$$

Hence to the conditions $B_1 = b_2$, $B_0 B_2 = b_0 b_2$ in § 5 for the equivalence of $2xy$, q_2 with $2xy$, Q_2, we may add $B_0^\tau + B_2^\tau = b_0^\tau + b_2^\tau$. From the latter and $B_0^\tau B_2^\tau = b_0^\tau b_2^\tau$, we find that B_0^τ, B_2^τ must equal, in some order, b_0^τ, b_2^τ. Hence the algebraic invariants a, b, θ and the modular invariant K_τ fully differentiate all pairs of forms of which the first is reducible, but not a multiple of a perfect square.

For a complete set of canonical types in which $q_1 = 2xy$, we may give q_2 the forms in the following table, which shows the values of the above invariants:

q_2	$\lvert q_2 \rvert$	θ	K_τ
$2b_1 xy$	$-b_1^2$	$-2b_1$	0
$\mu x^2 + 2b_1 xy \quad (\mu = 1 \text{ or } \nu)$	$-b_1^2$	$-2b_1$	μ^τ
$x^2 + 2b_1 xy + b_2 y^2 \quad (b_2 \neq 0)$	$b_2 - b_1^2$	$-2b_1$	$1 + b_2^\tau$
$\nu x^2 + 2b_1 xy + \nu c^2 y^2 \quad (c \neq 0)$	$\nu^2 c^2 - b_1^2$	$-2b_1$	-2

where b_1, b_2, c are arbitrary, while ν is a fixed not-square. Obviously these pairs are differentiated by the given invariants, necessarily absolute in view of q_1.

12. Finally, for $q_1 = a_0 x^2$, $q_2 = b_0 x^2$, the theory in § 6 is readily completed invariantively for a finite field. In view of the absolute invariants

$$Q_a = -a_0^\tau, \quad K_1 = -\tau a_0^{\tau-1} b_0,$$

necessary conditions for equivalence are $A_0^\tau = a_0^\tau$, $A_0^{\tau-1} B_0 = a_0^{\tau-1} b_0$. Multiplying the latter by $A_0 a_0$ and applying the former, we get $a_0 B_0 = A_0 b_0$. Hence $B_0 : b_0 = A_0 : a_0 =$ square. These, together with conditions (5) on the algebraic invariants a, b, θ, were shown to be sufficient conditions for the equivalence of two such pairs of forms.

As canonical types, when $q_1 = a_0 x^2$, we may take

$$q_1 = a_0 x^2 \,(a_0 = 1 \text{ or } \nu), \qquad q_2 = b_0 x^2 + b_2 y^2 \,(b_2 = 1 \text{ or } \nu), \qquad 2xy, \text{ or } b_0 x^2.$$

For $q_1 \equiv 0$, the canonical forms of q_2 are given by § 7.

13. As a partial summary of our results, we may state the

THEOREM. *Within a finite field of order p^n, $p > 2$, two pairs of binary quadratic forms are equivalent under linear transformation if, and only if, the algebraic invariants a, b, θ of the one pair equal the products of those of the other pair by the same square and the (absolute) modular invariants Q_a, K_1, K_τ, $K_{2\tau}$ have the same values for the two pairs of forms.**

REDUCTION OF TWO QUADRATIC FORMS IN THE $GF[2^n]$; THEIR INVARIANTS.

14. Consider two quadratic forms with coefficients in the $GF[2^n]$,

$$q_1 = a_0 x^2 + a_1 xy + a_2 y^2, \quad q_2 = b_0 x^2 + b_1 xy + b_2 y^2. \tag{54}$$

Under transformation (34), these become forms with the coefficients

$$a_2' = a_2 + ta_1 + t^2 a_0, \quad b_2' = b_2 + tb_1 + t^2 b_0, \quad a_i' = a_i, \; b_i' = b_i \quad (i = 0, 1). \tag{55}$$

Transformation (35), which now merely interchanges x and y, gives rise to

$$(a_0 a_2)\,(b_0 b_2). \tag{56}$$

Obvious (relative) invariants are a_1, b_1 and the resultant

$$R = a_2^2 b_0^2 + b_2^2 a_0^2 + a_2 (a_0 b_1^2 + a_1 b_0 b_1) + b_2 (a_1^2 b_0 + a_0 a_1 b_1). \tag{57}$$

15. We are led naturally to an important invariant (58) of a quadratic

*These seven invariants do not, however, form a complete system; there exist invariants of odd weights I hope to take up this problem on another occasion. For $p = 2$, see §§ 31–35.

form q_1 by determining the necessary and sufficient condition for its irreducibility. First, let q_1 be irreducible in the $GF[2^n]$; then each $a_i \neq 0$. For

$$x = a_0^{-1/2} X, \quad y = a_0^{1/2} a_1^{-1} Y,$$

we have

$$q_1 = X^2 + XY + \gamma Y^2, \quad \gamma = a_0 a_2 / a_1^2 = a_0 a_2 a_1^{2^n-3} \text{ or } a_0 a_2 a_1,$$

according as $n > 1$ or $n = 1$. Let $X = \xi + t\eta$, $Y = \eta$. Then

$$q_1 = \xi^2 + \xi\eta + c\eta^2, \quad c = t^2 + t + \gamma.$$

The latter is solvable for t in the $GF[2^n]$ if, and only if

$$\chi(c) = \chi(\gamma), \quad \text{where} \quad \chi(s) = \sum_{i=0}^{n-1} s^{2^i}.$$

If $\chi(\gamma) = 0$, we could choose t to make $c = 0$, contrary to the irreducibility of q_1. But $\chi^2 = \chi$. Hence a necessary condition for the irreducibility of q_1 is $\chi(\gamma) = 1$. The condition is also sufficient; for, if q_1 vanishes for $X = rY$, r in the $GF[2^n]$, then $r^2 + r \equiv \gamma$, so that $\chi(\gamma) = 0$. *Hence* $a_0 x^2 + a_1 xy + a_2 y^2$ *is irreducible in the* $GF[2^n]$ *if, and only if,* $H_a = 1$, *where*

$$H_a = \sum_{i=0}^{n-1} (a_0 a_1^{2^n-3} a_2)^{2^i} \text{ if } n > 1, \quad H_a = a_0 a_1 a_2 \text{ if } n = 1. \tag{58}$$

This function is unaltered by transformations (39) and (56). We next show that it is unaltered by (55). If $n = 1$, then $t^2 \equiv t$, so that the increment to H_a under (55) is $t a_0 a_1 (a_1 + a_0) \equiv 0$. Next, let $n > 1$. Since

$$(r + s)^{2^i} \equiv r^{2^i} + s^{2^i} \pmod 2,$$

the increment to H_a is

$$\sum_{i=0}^{n-1} a_0^{2^i} a_1^{2^i(2^n-2)} t^{2^i} + \sum_{i=0}^{n-1} a_0^{2^{i+1}} a_1^{2^i(2^n-3)} t^{2^{i+1}}.$$

In the first sum the term given by $i = 0$ may be replaced by the summand for $i = n$. In the new first sum we replace i by $i + 1$ and obtain the second sum, since the exponent $2^{i+1}(2^n - 2)$ of a_1 may be replaced by $2^i(2^n - 1 + 2^n - 3)$ and hence by $2^i(2^n - 3)$. *Hence** H_a *is an absolute invariant of* q_1 *in the* $GF[2^n]$.

A further absolute invariant of q_1 analogous to (45), is

$$I_a = (a_0^m - 1)(a_1^m - 1)(a_2^m - 1) \quad (m = 2^n - 1). \tag{59}$$

The invariants a_1, H_a, I_a of q_1 are independent† (§ 20).

* Cf. *Transactions*, l. c., pp. 213–214.

† Cf. *ibid*, § 28; in the second table of § 26, K is a misprint for χ.

16. To obtain simultaneous invariants of the pair (54), we replace each a_i by $a_i + kb_i$ in an invariant of q_1. Those obtained from H_a are functions of a_1, b_1, H_a, H_b, R (§ 30). For I_a, we set

$$I_{a+kb} = I_a + \sum_{r=1}^{m} k^r V_r. \tag{60}$$

We shall study the invariants V_r directly from the preceding definition. We can, however, obtain their explicit expressions, noting that, as in (41), each binomial coefficient C_i^m is odd:

$$\begin{aligned} V_r = (a_0^m - 1)(a_1^m - 1)\, a_2^{m-r} b_2^r + (a_2^m + b_2^m - 1)\, C_r \\ + \sum_{l=1}^{r-1} a_2^{m-r+l}\, b_2^{r-l}\, C_l + \sum_{l=r+1}^{m} a_2^{l-r}\, b_2^{m-l+r}\, C_l, \end{aligned} \tag{61}$$

$$\begin{aligned} C_l = (a_0^m - 1)\, a_1^{m-l} b_1^l + (a_1^m + b_1^m - 1)\, a_0^{m-l} b_0^l \\ + \sum_{i=1}^{l-1} a_0^{m-i} b_0^i\, a_1^{m-l+i} b_1^{l-i} + \sum_{i=l+1}^{m} a_0^{m-i} b_0^i\, a_1^{i-l} b_1^{m-i+l}. \end{aligned} \tag{62}$$

Thus, for $n = 1$,

$$V_1 = (a_0 - 1)(a_1 - 1)\, b_2 + (a_2 + b_2 - 1)(b_0 b_1 + b_0 + b_1 + a_0 b_1 + a_1 b_0). \tag{61'}$$

17. We pass to the reduction of the pair of forms (54), of which q_1 is now assumed to be irreducible in the $GF[2^n]$. Applying the transformations defined at the beginning of § 15, we get

$$q_1 = \xi^2 + \xi\eta + c\eta^2, \quad q_2 = e\xi^2 + f\xi\eta + g\eta^2, \tag{63}$$

where

$$e = b_0/a_0, \quad f = b_1/a_1, \quad g = t^2 b_0/a_0 + t\, b_1/a_1 + b_2 a_0/a_1^2,$$

c being a fixed root of $\chi(c) = 1$, t a root of $t^2 + t + a_0 a_2/a_1^2 = c$. For further reductions we must apply one of the automorphs of (63_1):

$$A:\ \begin{pmatrix} \alpha & \tau\alpha + c\beta \\ \beta & \alpha + (\tau + 1)\beta \end{pmatrix}, \quad \alpha^2 + \alpha\beta + c\beta^2 = 1;\ \tau = 0 \text{ or } 1.$$

Since $|A| = 1$, A replaces q_2 by a form with the same f. Hence it suffices to normalize

$$q_2 + f q_1 = (r\xi + s\eta)^2, \quad r^2 = e + f, \quad s^2 = g + cf.$$

The eliminant of q_1 and $r\xi + s\eta$ is

$$E_{rs} = s^2 + sr + cr^2.$$

Since E^2 is the resultant of forms (63), E is absolutely invariant under A (a verification is given below). Now A replaces $r\xi + s\eta$ by $\rho\xi + \sigma\eta$,

$$\rho = r\alpha + s\beta, \quad \sigma = (r\tau + s)\alpha + \{ rc + s(\tau + 1) \}\beta.$$

The determinant of the coefficients of α and β equals E_{rs}. Thus

$$\alpha E_{rs} = \rho \{ r c + s(\tau + 1) \} + \sigma s, \quad \beta E_{rs} = \rho (r\tau + s) + \sigma r,$$
$$(\alpha^2 + \alpha\beta + c\beta^2) E_{rs}^2 = E_{rs} E_{\rho\sigma}.$$

Hence if $E_{rs} = E_{\rho\sigma} \neq 0$, there exists a transformation A of determinant unity which replaces $r\xi + s\eta$ by $\rho\xi + \sigma\eta$. Next, $E_{rs} = 0$ implies $r = s = 0$, in view of the irreducibility of q_1. Hence two pairs of forms (63), with the same root c of $\chi(c) = 1$, are equivalent if, and only if, they have the same f and equal resultants. To obtain canonical types, we may set $r = 0$; then $q_2 + f q_1 = E\eta^2$.

It follows* that two pairs (54) having $H_a = 1$ are equivalent if, and only if, the ratios $a_1^4 : b_1^4 : R$ are the same for each pair.

18. The necessary (§15) and sufficient conditions that q_1 shall be reducible to $\xi\eta$ are $H_a = 0$, $a_1 \neq 0$. To prove them sufficient, set

$$x = (1 + a_0 k)\xi + k\eta, \quad y = a_1^{-1}(a_0\xi + \eta).$$

Then

$$q_1 = a_0^2 l \xi^2 + \xi\eta + l\eta^2, \quad l = a_0 k^2 + k + a_2/a_1^2.$$

If $H_a = 0$, we can determine k in the $GF[2^n]$ to make $l = 0$. Indeed, if $a_0 = 0$, we take $k = a_2/a_1^2$. If $a_0 \neq 0$, set $t = a_0 k$; then $a_0 l = t^2 + t + a_0 a_2/a_1^2$. Hence, as in §15, t can be chosen to make $a_0 l = 0$. Under the above transformation,

$$q_2 = B_0\xi^2 + a_1^{-1} b_1 \xi\eta + B_2\eta^2, \quad B_2 = b_0 k^2 + a_1^{-1} b_1 k + a_1^{-2} b_2, \quad B_0 = a_0^2 B_2 + b_0 + a_1^{-1} a_0 b_1.$$

The resultant of $\xi\eta$ and q_2 is $R = B_0 B_2$. If $B_0 \neq 0$, we multiply ξ by $B_0^{-1/2}$, η by $B_0^{1/2}$ and obtain

$$q_1 = \xi\eta, \quad q_2 = \xi^2 + a_1^{-1} b_1 \xi\eta + R\eta^2.$$

The case $B_0 = 0$, $B_2 \neq 0$, is reduced to the preceding by interchanging ξ and η. Finally, let $B_0 = B_2 = 0$, necessary and sufficient conditions for which are $b_0 = a_1^{-1} a_0 b_1$, $b_2 = a_1^{-1} a_2 b_1$, as is directly evident or as may be verified by eliminating k between $l = 0$, $B_2 = 0$. Then $q_2 = a_1^{-1} b_1 q_1$.

The two canonical types obtained when $R = 0$ may be differentiated by the absolute invariant V_1. For $a_0 = a_2 = 0$, $a_1 = 1$,

$$V_1 = b_1 (b_0^m - 1)(b_2^m - 1),$$

by §21. If $b_0 = b_2 = 0$, $V_1 = b_1$; if $b_0 \neq 0$, $V_1 = 0$. For the special case $b_1 = 0$, we distinguish the pairs by the invariant I_b.

---

* The transformation used to reduce (54) to (63) was of determinant $1/a_1$. Hence the resultant E^2 of (63) equals $1/a_1^4$ times the resultant R of (54). It is not difficult to verify this directly, employing the above values of e, f, g, r, s.

19. Next, q_1 is reducible to ξ^2 if, and only if, $a_1 = 0$, $I_a = 0$, the latter showing that a_0 and a_2 are not both zero. To avoid a separation into cases, we apply the transformation

$$\xi = a_0^{2^{n-1}} x + a_2^{2^{n-1}} y, \quad \eta = a_2^{2^{n-1}-1} x + a_0^{2^{n-1}-1} (a_2^{2^{n-1}} - 1) y,$$

of determinant $a_0^m (a_2^m - 1) - a_2^m = 1$, by $I_a = 0$. Solving, we get

$$x = a_0^{2^{n-1}-1} (a_2^{2^{n-1}} - 1) \xi + a_2^{2^{n-1}} \eta, \quad y = a_2^{2^{n-1}-1} \xi + a_0^{2^{n-1}} \eta.$$

Hence

$$q_1 = \xi^2, \quad q_2 = \beta_0 \xi^2 + b_1 \xi \eta + \beta_2 \eta^2, \quad \beta_2^2 = R.$$

For $\xi = X$, $\eta = lX + kY \, (k \neq 0)$, we get

$$q_1 = X^2, \quad q_2 = BX^2 + k b_1 XY + k^2 R^{1/2} Y^2, \quad B \equiv \beta_0 + b_1 l + R^{1/2} l^2.$$

If $b_1 \neq 0$, $R = 0$, we take $l = \beta_0 / b_1$, $k = 1/b_1$, and have $q_2 = XY$.

If $b_1 = 0$, $R \neq 0$, we take $l = \beta_0^{1/2} R^{-1/4}$, $k = R^{-1/4}$, and have $q_2 = Y^2$.

If $b_1 \neq 0$, $R \neq 0$, set $\rho = R^{1/2}/b_1^2$, and take $k = 1/b_1$. Then

$$q_2 = BX^2 + XY + \rho Y^2, \quad \rho B = \rho \beta_0 + \rho b_1 l + (\rho b_1 l)^2, \quad \chi(\rho B) = \chi(\rho \beta_0).$$

According as $\chi(\rho\beta_0) = 0$ or 1, we may take $B = 0$ or a fixed root of $\chi(B\rho) = 1$. For q_2, H_b is $\chi(B\rho)$, so that the two cases are distinguished by the invariant H_b.

If $b_1 = R = 0$, the types $q_1 = X^2$, $q_2 = BX^2$, are differentiated by the invariant $V_1 = B$. In fact, by § 21, for $a_0 = 1$, $a_1 = a_2 = 0$,

$$V_1 = b_0 (b_1^m - 1)(b_2^m - 1).$$

20. Finally, q_1 vanishes identically if, and only if, $I_a = 1$. As to q_2, we note that the types for a single form have been distinguished invariantively in § 15, and in the opening lines of §§ 18–20. This fact is shown in the following table, which is given primarily for convenience in the computations below:

Case	Coefficients	a_1	I_a	H_a
A	$\begin{cases} a_0 = a_1 = 1, & a_2 = c, \quad \chi(c) = 1, \\ b_0 = b_1 = f, & b_2 = e + cf \end{cases}$	1	0	1
B	$a_0 = a_2 = 0, \quad a_1 = 1$	1	0	0
C	$a_0 = 1, \quad a_1 = a_2 = 0$	0	0	0
D	$a_0 = a_1 = a_2 = 0$	0	1	0

An inspection of the table shows that the invariants are independent: no one is a rational integral function of the other two (§ 23).

21. For each case A, .., D, we shall determine the value of I_{a+kb} from the definition* (59) and then compare the result with (60) to determine the value of V_r. We consider the simplest case first. For the binomial expansions, see (41).

$$\text{(D)}\quad I_{kb} = 1 + k^m (I_b + 1)\,;\quad V_r = 0\ (r < m),\quad V_m = I_b + 1.$$

$$\text{(C)}\quad I = \Big(\sum_{r=1}^{m} k^r b_0^r\Big)(k^m b_1^m - 1)(k^m b_2^m - 1) = \sum_{r=1}^{m} k^r b_0^r (b_1^m - 1)(b_2^m - 1),$$
$$V_r = b_0^r (b_1^m - 1)(b_2^m - 1).$$

$$\text{(B)}\quad I = \Big(\sum_{r=1}^{m} k^r b_1^r\Big)(k^m b_0^m - 1)(k^m b_2^m - 1),\quad V_r = b_1^r (b_0^m - 1)(b_2^m - 1).$$

$$\text{(A)}\quad I = [(1 + kf)^m - 1]^2 [\{ke + c(1 + kf)\}^m - 1]$$
$$= [(1 + kf)^m - 1][k^m e^m - 1],\ \text{since}\ (s^m - 1)s = 0,$$
$$= \Big(\sum_{r=1}^{m} k^r f^r\Big)(k^m e^m - 1) = \sum_{r=1}^{m} k^r f^r (e^m - 1),\quad V_r = f^r (e^m - 1).$$

In case (D), $V_1 = 0$ if $n > 1$, $V_1 = I_b + 1$ if $n = 1$ $(m = 2^n - 1)$. The properties of V_1 are essentially different in the cases $n > 1$, $n = 1$; likewise the relations between V_1 and the earlier invariants. This difficulty would be largely obviated by the use of V_m in place of V_1 as the fundamental new invariant. While V_m (like V_1) serves with the earlier invariants to completely characterize the various types of two quadratic forms (§ 23, Note), V_m does not, for $n > 1$ form with those invariants a complete set of independent invariants (§ 29), whereas V_1 is found to possess this important property. Since it is essential to preserve V_1 if $n > 1$, we shall to replace V_1 when $n = 1$ by a modified form Z_1, such that V_1 $(n > 1)$ and Z_1 have similar properties.

It will be seen that there results complete uniformity for every n in the, properties of the new invariant replacing V_1, its relations with the invariants a_1, H_a, .., and with the V_r, if we set

$$Z_r = V_r\ (r < m),\quad Z_m = V_m + I_a (I_b + 1), \tag{64}$$

for every n. For $n = 1$ (61′) gives

$$Z_1 = a_2 b_2 (a_0 + 1)(a_1 + 1)(b_0 + 1)(b_1 + 1)$$
$$+ (a_2 + b_2 + 1)\{(a_0 a_1 + a_0 + a_1)(b_0 b_1 + b_0 + b_1) + a_0 b_1 + a_1 b_0\}. \tag{64′}$$

* We may also use (61)–(62). In case (A), each $C_i = f^i$.

The above results and those for I_a in § 20 give

$$\left.\begin{array}{l}(\mathrm{A})\ Z_r = f^r(e^m - 1);\ (\mathrm{B})\ Z_r = b_1^r(b_0^m - 1)(b_2^m - 1);\\ (\mathrm{C})\ Z_r = b_0^r(b_1^m - 1)(b_2^m - 1);\ (\mathrm{D})\ Z_r = 0\end{array}\right\}(r = 1, \ldots, m). \tag{65}$$

Since $\sigma^2 \equiv \sigma$, $\sigma^r \equiv \sigma$ if $\sigma = s^m - 1$, we have*

$$Z_r = Z_1^r \tag{66}$$

for every set of values of the a_i in the field. Hence (66) is a formal equality.

The importance of Z_1 lies in the following interpretation. If q_1 is not identically zero, $Z_1 = t$ if $q_2 \equiv t q_1$, $Z_1 = 0$ if q_2/q_1 is not a constant. If $q_1 \equiv 0$, then $Z_1 = 0$.

22. The following table gives a complete set of non-equivalent canonical types (§§ 17–20) of pairs of quadratic forms in the $GF[2^n]$, and the values for each pair of a set of invariants completely characterizing the types:

	q_1	q_2	a_1	b_1	I_a	I_b	H_a	H_b	R	Z_1
I	$x^2 + xy + cy^2$	$fx^2 + fxy + (e + cf)y^2$	1	f	0	π	1	χ_1	e^2	$f(e^m - 1)$
II	xy	$x^2 + fxy + Ry^2$	1	f	0	0	0	χ_2	R	0
III	xy	fxy	1	f	0	$f^m - 1$	0	0	0	f
IV	x^2	$Bx^2 + xy + \rho y^2$	0	1	0	0	0	1	ρ^2	0
V	x^2	$xy + \sigma y^2$	0	1	0	0	0	0	σ^2	0
VI	x^2	y^2	0	0	0	0	0	0	1	0
VII	x^2	$b_0 x^2$	0	0	0	$b_0^m - 1$	0	0	0	b_0
VIII	0	$x^2 + xy + cy^2$	0	1	1	0	0	1	0	0
IX	0	xy	0	1	1	0	0	0	0	0
X	0	x^2	0	0	1	0	0	0	0	0
XI	0	0	0	0	1	1	0	0	0	0

Here $\chi(s) = \sum_{i=0}^{n-1} s^{2^i}$, $m = 2^n - 1$, c and B are particular solutions of $\chi(c) = 1$, $\chi(B\rho) = 1$, while $\rho \neq 0$. In the following abbreviations

$$\pi = (f^m - 1)(e^m - 1), \quad \chi_1 = f^m + \chi(f^{2^n-2} e), \quad \chi_2 = \chi(f^{2^n-2} R), \tag{67}$$

the exponents are to be replaced by unity when $n = 1$.

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*Special cases may be seen by inspection from (60), since $I^2 \equiv I$. Thus

$$V_r^2 = V_{2r},\ V_{2^{n-1}+\rho}^2 = V_{2\rho+1}\ (\rho = 0;\ r, \rho = 1, 2, \ldots, 2^{n-1} - 1).$$

The same relations hold between the C's in (62). In fact, a_2^m enters (61) only with the coefficient C_r; hence the coefficient of a_2^m in V_r^2 is C_r^2.

23. THEOREM. *The invariants a_1, b_1, I_a, I_b, H_a, H_b, R, Z_1 of a pair of quadratic forms in the $GF[2^n]$ are independent: no one is a rational integral function of the others with coefficients in the field.*

To prove that a given invariant is independent of the others, it suffices to specify two pairs of forms for which the given invariant has distinct values, while each of the remaining invariants have the same value for the two pairs of forms. These requirements may be met as follows:

a_1: II, V, $f=1$, $R=\sigma=0$;	H_a: I, III, $e=f=0$;
b_1: V, VI, $\sigma=1$;	H_b: $VIII$, IX;
I_a: VII, XI, $b_0=0$;	R : V, $\sigma=0$, $\sigma=1$;
I_b: II, III, $f=R=0$;	Z_1: II, III, $f=1$, $R=0$.

Note. In view of (66), the proof holds true if we replace Z_1 by any Z_r.

24. Of the preceding eight invariants, a_1, b_1, R are relative, the remaining five absolute. In the proof in § 23, a_1, b_1, R each had the values 0, 1 (and hence their ratio is not a power of the determinant of transformation), when the other seven invariants were equal. Hence all eight invariants are necessary to characterize the canonical forms; in §§ 17–20 they were shown to be sufficient.

THEOREM. *Two pairs of quadratic forms in the $GF[2^n]$ are equivalent if and only if the ratios $a_1^4 : b_1^4 : R$ and the absolute invariants I_a, I_b, H_a, H_b, Z_1 have the same values for each pair of forms.*

25. We shall establish certain important relations between the invariants by verifying the relation for each set of values a_i, b_i defining types I–XI of § 22, and by noting that the relation continues valid when a_1 and b_1 are multiplied by Δ, R by Δ^4, where Δ is any mark $\neq 0$, so that $\Delta^m=1$. The relation will then be true for every set a_i, b_i in the field and hence be an identity. We begin with

$$\chi[(a_1 b_1)^{2^n-3}R]=a_1^m H_b+b_1^m H_a,\quad H_a(R^m+I_b+1)=H_b(R^m+I_a+1),\quad (68)$$

in which 2^n-3 is to be replaced by unity if $n=1$. We have $H_a=0$ except for I; $a_1 R=a_1 H_b=0$, except for I and II. Hence proof of (68_1) is needed only for I and II, when it reduces to (67_2), (67_3). Note that $\chi=\chi^2$, so that

$$\chi(f^{2^n-2}e)=\chi(f^{2^n-1+2^n-3}e^2)=\chi(f^{2^n-3}e^2).$$

As to (68_2), $H_a = 0$ except for *I*, $H_b = 0$ except for *I, II, IV, VIII.* Taking these cases in turn, we give the relation to which (68) reduces and then its proof:

(I) $e^m + \pi + 1 = (e^m + 1)\chi_1$; each $= (e^m - 1) f^m$, since $e(e^m - 1) = 0$.

(II) $0 = (R^m + 1)\chi_2$, since $(R^m + 1) R = 0$.

(IV) $0 = \rho^m + 1$, since $\rho \neq 0$.

(VIII) $0 = 1 + 1$ (mod. 2).

By a similar argument we readily prove that

$$Z_1^m = (a_1^m - 1)(b_1^m - 1)\{R^m + (I_a - 1)(I_b - 1)\} + a_1 b_1^{m-1} Z_1, \qquad (69)$$

$$I_a Z_1 = I_b Z_1 = R Z_1 = 0, \quad H_a Z_1 = H_b Z_1 = a_1^{m-1} b_1 H_a (R^m - 1). \qquad (70)$$

From the definitions of the invariants, we have by inspection

$$a_1 I_a = b_1 I_b = R I_a = R I_b = H_a I_a = H_b I_b = 0, \qquad (71)$$

$$a_1^m H_a = H_a = H_a^2, \quad b_1^m H_b = H_b = H_b^2, \quad I_a^2 = I_a, \quad I_b^2 = I_b, \qquad (72)$$

while, of course, for any invariant k,

$$k^{m+1} \equiv k^{2^n} = k. \qquad (73)$$

26. Other needed relations will be derived from (68)–(73). Multiplying (69) by $b_1 Z_1^s$, we get $b_1 Z_1^s = a_1 b_1^m Z_1^{s+1}$. The case $s = m$ shows that

$$b_1 Z_1^k = a_1^{m+1-k} b_1^k Z_1 \qquad (k = 1, \ldots, m) \qquad (74)$$

is true when $k = m$. We prove (74) by induction from k to $k - 1$:

$$b_1 Z_1^{k-1} = a_1 b_1^m Z_1^k = a_1^{m+1-(k-1)} b_1^{k-1} Z_1.$$

Similarly, we multiply (69) by $a_1 Z_1^s$ and prove that, if $n > 1$,

$$a_1 Z_1^k = a_1^{m+2-k} b_1^{m+k-1} Z_1 \qquad (k = 1, \ldots, m), \qquad (75)$$

in which we may suppress the m in the exponent of b_1 if $k > 1$, that of a_1 if $k = 1$. By (74) and (75) for $k = 1$, we get, if $n > 1$,

$$b_1^m Z_1 = b_1^{m-1}(a_1^m b_1 Z_1) = a_1^m b_1^m Z_1 = a_1^{m-1}(a_1 b_1^m Z_1) = a_1^m Z_1. \qquad (76)$$

This result follows at once for any n from the table in § 22. By (73)–(76), every product containing Z_1 and formed from a_1, b_1, Z_1 can be reduced to

$$Z_1^2, \; Z_1^3, \; \ldots, \; Z_1^m, \; a_1^m Z_1, \; a_1^i b_1^j Z_1 \qquad (i, j = 0, 1, \ldots, m - 1). \qquad (77)$$

27. For $n > 1$, we multiply (68_1) by $a_1 b_1$; then R^{2^i} has the coefficient $(a_1 b_1)^k$, where

$$k = (2^n - 3) 2^i + 1 = (2^n - 1)(2^i - 1) + 2^n - 2^{i+1},$$

the first part of which may be suppressed by (73) if $i < n - 1$; while if

$i = n-1$, we may reduce k to $2^n - 1 = m$. We multiply the resulting relation by H_a and apply (72_1), by H_b and apply (72_2), and get

$$b_1^m H_a R^{2^{n-1}} = H_a \sum_{i=0}^{n-2} (a_1 b_1)^{2^n - 2^{i+1}} R^{2^i} + a_1 b_1 H_a + a_1 b_1 H_a H_b, \tag{78}$$

$$a_1^m H_b R^{2^{n-1}} = H_b \Sigma \qquad\qquad\qquad\qquad + a_1 b_1 H_b + a_1 b_1 H_a H_b. \tag{79}$$

Multiplying (78) by H_b, or (79) by H_a, we get

$$H_a H_b R^{2^{n-1}} = H_a H_b \sum_{i=0}^{n-2} (a_1 b_1)^{2^n - 2^{i+1}} R^{2^i}. \tag{80}$$

28. We proceed to reduce as far as possible the exponent k of R in a product formed from a_1, b_1, H_a, H_b, R, in which initially $k > 0$. First, let R^m occur. We first eliminate the terms involving $H_b R^m$ by (68_2). Multiplying the latter by a_1^m, and applying (72_1), (79), we have $H_a R^m$ expressed in terms of R^t $(t < m)$. By (68_1), $a_1 b_1 R^m$ is a function of the R^t $(t < m)$. Hence the coefficient of R^m may be assumed to be a linear combination of the a_1^i, b_1^i.

Next, consider a product involving $R^k (2^{n-1} \leqq k < m)$. By (79), (78), (68_1), (80), a term with a factor $a_1 H_b R^k$, $b_1 H_a R^k$, $a_1 b_1 R^k$, or $H_a H_b R^k$, may be expressed in terms of R^t $(t < k)$. Hence the coefficient of R^k may be assumed to be a linear combination of $b_1^i H_b$, $a_1^i H_a$, a_1^i, b_1^i.

Let the same reductions be effected in the terms involving R^m in (69) and (70). In (77), we suppress Z_1^m if $n > 1$, but $a_1^m Z_1$ if $n = 1$, by means of the reduced form of (69). Employing also (70)–(73), we have at once the following

THEOREM. *Any rational integral function of the invariants*

$$a_1,\ b_1,\ I_a,\ I_b,\ H_a,\ H_b,\ R,\ Z_1 \tag{81}$$

may be reduced by means of relations (68)–(73), *together with* (76) *for* $n = 1$, *to a linear function of*

$$\left.\begin{array}{l} I_a I_b,\ a_1^r I_b,\ b_1^r I_a,\ a_1^i H_a I_b,\ b_1^i H_b I_a, \\ a_1^r b_1^s,\ a_1^i b_1^r H_a,\ b_1^i a_1^r H_b,\ a_1^i b_1^j H_a H_b, \\ a_1^r b_1^s R^c,\ a_1^i b_1^r H_a R^c,\ b_1^i a_1^r H_b R^c,\ a_1^i b_1^j H_a H_b R^c \quad (c = 1, \ldots, 2^{n-1} - 1), \\ a_1^r R^d,\ b_1^{i+1} R^d,\ a_1^i H_a R^d,\ b_1^i H_b R^d \quad (d = 2^{n-1}, \ldots, m-1), \\ a_1^r R^m,\ b_1^{i+1} R^m,\ a_1^i b_1^j Z_1,\ a_1^m Z_1,\ Z_1^2,\ Z_1^3,\ \ldots,\ Z_1^{m-1}, \end{array}\right\} \tag{82}$$

where $r, s = 0, 1, \ldots, m$; $i, j = 0, \ldots, m-1$; $m \equiv 2^n - 1$, *the invariant* $a_1^m Z_1$ *being suppressed if* $n = 1$.

For $n=1$, the third and fourth lines of (82) are missing, while the last line includes only

$$R, \quad a_1 R, \quad b_1 R, \quad Z_1; \tag{82'}$$

and the final relations (70) may be reduced by (68_1) to

$$H_a Z_1 = H_b Z_1 = H_a H_b. \tag{70'}$$

29. For $n=1, 2, 3$, we prove below that every invariant of the pair of forms (54) is an integral function of the eight invariants (81), which thus form a complete system. The proof is so conducted as to show incidentally that the invariants (82) are linearly independent. The latter thus form a complete set of linearly independent invariants of the pair of forms.

Although the first seven invariants (81), together with $Z_m = Z_1^m$, completely characterize the various canonical types of two quadratic forms, they do not form a complete sytem of independent invariants. In fact, every product involving Z_m reduces, in view of (70), to $a_1^i b_1^j Z_m$ and a function of $R, \ldots$. We may restrict i and j to values $< m$. For, by multiplying (69) by $a_1^m - 1$ and by $b_1^m - 1$, we see that $a_1^m Z_m$ and $b_1^m Z_m$ equal Z_m plus a function of $R, \ldots$. Hence the present list of linearly independent invariants is now smaller than (82), lacking terms corresponding to $a_1^m Z_1$, $Z_1^2, \ldots, Z_1^{m-1}$.

30. The method of obtaining simultaneous invariants from the invariants of a single form by replacing each a_i by $a_i + k b_i$ was applied in § 16 to I_a, but not to H_a. Let

$$H_{a+kb} = H_a + k^m H_b + \sum_{i=1}^{m} k^i S_i \qquad (m = 2^n - 1). \tag{83}$$

For use in § 31, we note that when $n=1$,

$$S_1 = a_2(a_0 b_1 + a_1 b_0 + b_0 b_1) + b_2(a_0 b_1 + a_1 b_0 + a_0 a_1). \tag{84}$$

If we interchange the a's and b's, replace k by k^{m-1}, and multiply the result by $k\, k^{2^n-3} k = k^m$, we obtain the same expression as when we merely multiply a by k^m. Hence S_i and S_{m-i} are permuted by interchanging the a's and b's. Again, since $H^2 = H$,

$$S_i^2 = S_{2i}, \quad S_{2^{n-1}+i}^2 = S_{2i+1} \qquad (i < 2^{n-1}). \tag{85}$$

In view of the two results, every S_i may be obtained at once from S_1 if $n \leq 3$

from S_1, S_3, S_5 if* $n=4$ or 5. For any n, we shall determine the values of S_1, and for $n>3$ those of S_3, and S_5 for the four cases enumerated in § 20.

$$\text{(D)} \quad H_{kb}=k^m H_b, \text{ each } S_i=0.$$

$$\text{(C)} \quad H=k^m H_b+\sum_{i=0}^{n-1}(k^{m-1}b_1^{m-2}b_2)^{2^i},$$

the exponents of k and b_1 being replaced by unity if $n=1$, so that $S_1=b_1 b_2$. For $n>1$, the only non-vanishing S_i are obviously

$$S_{m-2^i}=b_1^{m-2^{i+1}}b_2^{2^i} \qquad (i=0,\,1,\,..,\,n-1).$$

Thus $S_1=b_1^2 b_2^2$ if $n=2$, $S_1=0$ if $n>2$, $S_3=S_5=0$ if $n>3$.

$$\text{(B)} \quad H=k^m H_b+\sum_{i=0}^{n-1}\Big[\sum_{j=0}^{m-3}C_j^{m-2}k^{j+2}b_0 b_2 b_1^j\Big]^{2^i} \qquad (n>1).$$

If $n=1$, $S_1=b_0 b_2$. The binomial coefficient C_j^{m-2} is odd if, and only if, $j=4l$ or $4l+1$. Now $x\,2^i\equiv 1\equiv 2^n \pmod m$ gives $x\equiv 2^{n-i}$; thus $j+2=2^{n-i}$ makes C_j^{m-2} even unless $n-i=1$, $j=0$. Hence $S_1=(b_0 b_2)^{2^{n-1}}$ for every n. To determine S_3 for $n>2$, we note that $x\,2^i\equiv 3 \pmod m$ gives $x\equiv 3\,.\,2^{n-i}$. For $i>1$, $j+2=3\,.\,2^{n-i}$ makes C_j^{m-2} even unless $n-i=1$, $j=4$. For $i=1$, $j+2=2^{n-1}+1$ makes C_j even. For $i=0$, $j=1$. Hence $S_3=(b_0 b_2 b_1^4)^{2^{n-1}}+b_0 b_2 b_1$. Similarly, the four possible cases for i give

$$S_5=(b_0 b_2 b_1^8)^{2^{n-1}}+(b_0 b_2 b_1^{2^{n-1}})^2,\ n>3.$$

$$\begin{aligned}\text{(A)} \quad H&=\sum_{i=0}^{n-1}\big[(1+kf)^{m-1}\{ke+c(1+kf)\}\big]^{2^i}\\ &=(1+kf)^m\Sigma c^{2^i}+\Sigma\big[(1+kf)^{m-1}ke\big]^{2^i}\\ &=\sum_{s=0}^{m}k^s f^s+\sum_{i=0}^{n-1}\Big[\sum_{j=0}^{m-1}(j+1)k^{j+1}f^j e\Big]^{2^i},\end{aligned}$$

since $\Sigma c^{2^i}=1$, $C_s^m\equiv 1$, $C_j^{m-1}\equiv j+1 \pmod 2$. The terms of S_t have $j+1\equiv t\,2^{n-i} \pmod m$. As above, we find that†

$$S_1=f+e\ (n>1),\quad S_3=f^3+f^2e+fe^2,\quad S_5=f^5+f^4e+fe^4\ (n>2),$$

while by a special examination, $S_1=e$, if $n=1$.

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* If $n=6$, we would need S_1, S_3, S_5, S_7, S_9, S_{11}.

† $S_7=f^7+f^6e+f^5e^2+f^3e^4$. Note that $S_3=H_b$ for $n=2$, $S_7=H_b$ for $n=3$.

We deduce the identities

$$\left.\begin{array}{l} S_1 = (a_1 + b_1 + a_1 b_1) R \; (n = 1), \quad S_1 = (a_1^3 - 1) b_1^2 R + a_1 R^2 + a_1^2 b_1 H_a \; (n = 2), \\ S_1 = a_1^{2^n-3} R^{2^{n-1}} + a_1^{2^n-2} b_1 H_a \; (n > 2), \\ S_3 = a_1^{2^n-5} b_1^2 R^{2^{n-1}} + a_1^{2^n-6} b_1 R + a_1^{2^n-4} b_1^3 H_a \; (n > 3), \\ S_5 = a_1^{2^n-7} b_1^4 R^{2^{n-1}} + a_1^{2^n-6} b_1 R^2 + a_1^{2^n-6} b_1^5 H_a \; (n > 3), \end{array}\right\} \qquad (86)$$

with the additional terms $(a_1^7 - 1) b_1^6 R^2$ in S_3 if $n = 3$.

DETERMINATION OF ALL THE INVARIANTS IN THE $GF[2^n]$, $n \leqq 3$.

31. Let $n = 1$, so that the a_i, b_i are integers modulo 2. Then

$$I_a + H_a + a_1 + 1 = J_a = a_2 (a_0 + a_1 + 1) + a_0 a_1 + a_0 , \qquad (87)$$

$$V_1 + S_1 + I_b + 1 = \sigma = a_2 (b_0 + b_1) + b_2 (a_0 + a_1) + a_0 b_1 + a_1 b_0 \qquad (88)$$

are invariants of the second degree defined by the earlier invariants (58), (59), (61′), (84). We may also derive σ from $J_{a+kb} = J_a + k J_b + k\sigma$.

Any integral function of the a_i, b_i may be given the form

$$\phi = E a_2 b_2 + F a_2 + G b_2 + K \quad (E, \ldots, \text{functions of } a_0, a_1, b_0, b_1).$$

Under the substitution (55), with $t = 1$, ϕ takes the increment

$$a_2 E (b_1 + b_0) + b_2 E (a_1 + a_0) + \{ E (a_1 + a_0)(b_1 + b_0) + F (a_1 + a_0) + G (b_1 + b_0) \}.$$

If ϕ is invariant the three parts must be zero (mod 2). Hence

$$E = e_1 (1 + b_1 + b_0) + e_2 b_0 b_1 , \quad e_i = s_i (1 + a_1 + a_0) + t_i a_0 a_1 ,$$
$$F (a_1 + a_0) = G (b_1 + b_0),$$

where s_i and t_i are constants. Hence the invariant

$$\phi' = \phi - t_2 H_a H_b - s_2 J_a H_b - t_1 H_a J_b - s_1 J_a J_b$$

has $E' = 0$. Then by (56), no term of ϕ' has a factor $a_2 b_2$ or $a_0 b_0$. This property is true of the following invariants:

$$J_a, \quad \sigma, \quad a_1 b_1 \sigma, \quad b_1 \sigma, \quad a_1 \sigma, \quad b_1 (J_a + \sigma), \quad H_a, \quad b_1 H_a ,$$

in which the coefficient of a_2 has the respective values

$$a_0 + a_1 + 1, \quad b_0 + b_1 , \quad a_1 b_0 b_1 + a_1 b_1 , \quad b_0 b_1 + b_1 , \quad a_1 b_0 + a_1 b_1 ,$$
$$a_0 b_1 + a_1 b_1 + b_0 b_1 , \quad a_0 a_1 , \quad a_0 a_1 b_1 .$$

Subtracting constant multiples of the preceding invariants from ϕ', we obtain an invariant ϕ_1 having $E_1 = 0$, and such that F_1 lacks the terms

$$1, \quad b_0 , \quad a_1 b_0 b_1 , \quad b_1 , \quad a_1 b_0 , \quad a_1 b_1 , \quad a_0 a_1 , \quad a_0 a_1 b_1 ,$$

no one of which occurs in a later one of the above combinations. Then

$$F_1 = c\,a_0 + d\,a_1 + e\,a_0 b_1 + f\,b_0 b_1 .$$

But $F_1(a_1 + a_0)$ must vanish when $b_1 = b_0$. Thus $F_1 \equiv 0$. Hence no term of ϕ_1 contains a_2 or a_0. Thus $\phi_1 = \beta_1 + a_1 \beta_2$, where the β's are functions of the b_i only. But a_1 is an invariant. Hence β_1 and β_2 must be invariants. But the above discussion shows that every invariant involving only the a_i is a linear function of J_a, H_a, a_1. Hence the β's are linear functions of J_b, H_b, b_1.

THEOREM. *Every invariant of a pair of binary quadratic forms modulo 2 is an integral function of σ and the invariants of the separate forms; every invariant is a linear function of the following twenty:*

$$H_a H_b,\ H_a J_b,\ H_b J_a,\ J_a J_b,\ a_1^i H_b,\ b_1^i H_a,\ a_1^i J_b,\ b_1^i J_a,\ a_1^i b_1^j,\ a_1^i b_1^j \sigma \qquad (i, j = 0, 1). \tag{89}$$

If we eliminate $H_a = a_1 J_a$ and $H_b = b_1 J_b$, we obtain

$$a_1^i b_1^j (1,\ \sigma,\ J_a,\ J_b,\ J_a J_b) \qquad (i, j = 0, 1). \tag{89$'$}$$

The product of any two invariants (89′) can be reduced to a linear function of the same by use of the relations

$$\sigma J_a = a_1 \sigma + \sigma, \quad \sigma J_b = b_1 \sigma + \sigma, \tag{90}$$

and $a_1^2 = a_1$, etc. For the resultant of the forms, we have (end of § 32)

$$R = (1 + a_1 b_1)\sigma + b_1 J_a + a_1 J_b . \tag{91}$$

Then (86_1), (93), (87), (88), (64) give the other invariants in terms of (89′).

In the notations of § 28, it now follows for $n = 1$ that every invariant is an integral functions of the eight invariants (81), and that the twenty invariants given by (82′) and the first two lines of (82) form a complete set of linearly independent invariants.

32. For any n we set, in generalization of (87),

$$J_a = I_a + H_a + a_1^m - 1, \quad J_b = I_b + H_b + b_1^m - 1 \; (m = 2^n - 1). \tag{92}$$

Then, by (71) and (72),

$$H_a = a_1^m J_a,\ H_b = b_1^m J_b,\ I_a = (a_1^m - 1)(J_a - 1),\ I_b = (b_1^m - 1)(J_b - 1). \tag{93}$$

The invariants of a single form may therefore be expressed in terms of two. Hence the eight invariants (81) may be expressed in terms of six. For $n = 1$, we expressed (in § 31) all the invariants in terms of the invariants of the single forms and one additional invariant σ. But, for $n > 2$, there exists no combination C of the S_1 and invariants (81), other than R, in terms of which R can

be expressed rationally. Indeed, for the two pairs of forms under V in § 22, with $\sigma=0$ and $\sigma=1$, respectively, the invariants (81), other than R, take the same value, while $S_1=0$ by (86). When $n=1$ or 2, $S_1=\sigma$ or σ^2 for forms V. The exceptional nature of the case $n=1$ is due to the relation (86): $R=S_1+(a_1-1)(b_1-1)R$ which, by (69), enables us to express R in terms of S_1, Z_1, a_1, b_1, I_a, I_b. For $n=2$, (86) gives

$$(S_1-a_1^2 b_1 H_a)^3=(a_1^3 b_1^3+a_1^3+b_1^3)R^3,$$

so that, by (69), R^3 can be expressed in terms of S_1, Z_1, etc.

33. Next, let $n=2$. Let the general polynomial

$$\phi=\sum_{i,j}^{0,1,2,3} D_{ij}\,a_2^i\,b_2^j \qquad (D\text{'s functions of } a_0,\ a_1,\ b_0,\ b_1)$$

become ϕ' under transformation (55). The coefficient of t in $\phi'-\phi$ is

$$a_1\phi_{a_2}+b_1\phi_{b_2}+a_0^2(\tfrac{1}{2}\phi_{a_2^2})+b_0^2(\tfrac{1}{2}\phi_{b_2^2})+a_0 b_0\phi_{a_2 b_2}+\Sigma E_{rs}\frac{1}{r!\,s!}\phi_{a_2^r b_2^s}, \tag{94}$$

with $r+s\geq 3$, the values of the E_{rs} not being required in the treatment here employed. The divisons by 2, $r!\,s!$ are to be performed algebraically and the quotients alone interpreted in the $GF[2^2]$. A second* annihilator of an invariant ϕ is given by the coefficient of t^2 in $\phi'-\phi$; it may be obtained from (94) by applying the substitution $(a_0 a_1)(b_0 b_1)$, as follows from (55) for $n=2$. We shall designate by (k') the relation derived by applying $(a_0 a_1)(b_0 b_1)$ to a relation (k) deduced from (94).

The coefficients of $a_2^2 b_2^3$, $a_2^3 b_2^2$, $a_2 b_2^3$, $a_2^3 b_2$ in (94) give

$$a_1 D_{33}=b_1 D_{33}=a_0^2 D_{33}=b_0^2 D_{33}=0.$$

Hence

$$D_{33}=c\,(a_0^3-1)(a_1^3-1)(b_0^3-1)(b_1^3-1).$$

After subtracting $c\,I_a I_b$ from ϕ, we have $D_{33}=0$. Then, by (56), no term of ϕ has a factor $a_0^3 b_0^3$. For $D_{33}=0$, the coefficients of $a_2^2 b_2^2$, $a_2^2 b_2$, $a_2 b_2^2$, a_2^3, b_2^3, $a_2 b_2$ in (94) give

$$a_1 D_{32}=b_1 D_{23},\quad a_1 D_{31}=b_0^2 D_{23},\quad b_1 D_{13}=a_0^2 D_{32}, \tag{95}$$

$$b_1 D_{31}=b_0^2 D_{32},\quad a_1 D_{13}=a_0^2 D_{23},\quad a_0^2 D_{31}=b_0^2 D_{13}. \tag{96}$$

Let δ_{ij} be the coefficient of $b_0^i b_1^j$ in D_{32}. By $b_0\,(96_1)+b_1\,(96_1')$,

$$(b_0^3+b_1^3)D_{32}=0,\quad \delta_{10}=\delta_{20}=\delta_{01}=\delta_{02}=0,\quad \delta_{03}=\delta_{30}=\delta_{00}.$$

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* That given by t^3 is a consequence of the other two.

By (95_3), $a_0^2 D_{33}$ is a multiple of b_1. Hence

$$a_0^2 \delta_{00} (1 + b_0^3) = 0,$$

so that δ_{00} has the factor $a_0^3 - 1$. But in ϕ, $\delta_{00} = \delta_{30}$ is multiplied by b_0^3. Since a factor $a_0^3 b_0^3$ can not occur, $\delta_{00} = 0$. Hence

$$D_{33} = \Sigma\, \delta_{ij} b_0^i b_1^j \qquad (i, j = 1, 2, 3). \tag{97}$$

By $b_0(95_1) + b_1(95_1')$, we have $(a_0 b_1 + a_1 b_0) D_{33} = 0$. Hence

$$a_0 \delta_{ij-1} = a_1 \delta_{i-1j} \qquad (i, j = 1, 2, 3), \tag{98}$$

in which a subscript 0 is to be replaced by 3; note that in (97) each subscript take distinct values modulo 3. Since δ_{3j} is free of a_0^3, (98), for $i = 3$, requires that $a_0 \delta_{3j-1}$ and hence each δ_{3j} be a multiple of a_1. Then by (98), for $i = 1$, $a_1 \delta_{3j}$, and hence also δ_{3j}, is a multiple of a_0. Thus

$$\delta_{3j} = a_0^2 \sum_{k=1}^{3} c_{jk} a_1^k + a_0 \sum_{k=1}^{3} d_{jk} a_1^k \qquad (j = 1, 2, 3),$$

the c's and d's being constants whose subscripts may be reduced modulo 3 without causing ambiquity. Then by (98), for $i = 3$,

$$\delta_{2j} = a_0^3 \sum_{k=1}^{3} c_{j-1k} a_1^{k-1} + a_0^2 \sum_{k=1}^{3} d_{j-1k} a_1^{k-1} + (a_1^3 - 1) \sum_{i=0}^{3} \rho_{ji} a_0^i.$$

Now $a_1 \delta_{1j} = a_0 \delta_{2j-1}$, so that the latter has no terms free of a_1. Thus

$$\rho_{s1} = 0, \quad \rho_{s2} = d_{s-11}, \quad \rho_{s3} = \rho_{s0} + c_{s-11} \quad (s = 1, 2, 3),$$

$$\delta_{2j} = a_0^3 \sum_{k=2}^{4} c_{j-1k} a_1^{k-1} + a_0^2 \sum_{k=3}^{4} d_{j-1k} a_1^{k-1} + \rho_{j0} (a_0^3 - 1)(a_1^3 - 1).$$

Now the coefficients of $a_2^3 b_2^3$ in $H_b I_a$ and $H_a H_b R$ are

$$(a_0^3 - 1)(a_1^3 - 1) b_0^3 b_1^2, \quad a_0 a_1 b_0 b_1^3 + a_0^2 a_1^3 b_0^3 b_1^3 + a_0^3 a_1^2 b_0^2 b_1.$$

Hence by subtracting from ϕ constant multiples of

$$b_1^i H_b I_a, \qquad a_1^i b_1^j H_a H_b R \qquad (i, j = 0, 1, 2),$$

we may delete from D_{33} the terms $a_0^3 b_0^3 b_1^r$, $a_0^3 b_0^2 a_1^s b_1^r$ $(s, r = 1, 2, 3)$. Then each δ_{3r} is free of a_0^3, so that each $c_{st} = 0$, $\rho_{s0} = 0$. In the simplified form of δ_{2j}, let $k = l + 1$, then

$$\delta_{3j} = a_0 \sum_{k=1}^{3} d_{jk} a_1^k, \quad \delta_{2j} = a_0^2 \sum_{l=1}^{3} d_{j-1\,l+1} a_1^l \qquad (j = 1, 2, 3).$$

By (98), for $i = 2$, $a_1 \delta_{1j} = a_0 \delta_{2j-1}$, so that

$$\delta_{1j} = a_0^3 \sum_{l=1}^{3} d_{j-2\,l+1} a_1^{l-1} + (a_1^3 - 1) A_j' \equiv a_0^3 \sum_{l=3}^{4} d_{j-2\,l+1} a_1^{l-1} + (a_1^3 - 1) A_j,$$

where A' and A are functions of a_0. Replacing l by $l+1$ and applying (98) for $i=1$, we see that $a_0 A'_{j-1}=0$, so that, by (97),

$$D_{33}=\sum_{j=1}^{3} b_1^j \{b_0^8 a_0 \sum_{l=1}^{3} d_{jl} a_1^l + b_0^2 a_0^2 \sum_{l=1}^{3} d_{j-1\, l+1} a_1^l + b_0 a_0^3 \sum_{l=1}^{3} d_{j-2\, l+2} a_1^l + b_0 \sigma_j A\},$$

where $A=(a_0^8-1)(a_1^8-1)$, and the σ_j are constants. Then by (95_3),

$$D_{13}=\sum_{j=1}^{3} b_1^{j+2} \{b_0^8 a_0^8 \Sigma + b_0^2 a_0 \Sigma + b_0 a_0^2 \Sigma\} + (b_1^8-1)\, G,$$

the sums being the same as in D_{32}. By $a_0^2(95_3') + a_1^2(95_3)$,

$$(a_0^2 b_0 + a_1^2 b_1)\, D_{13}=0.$$

Hence $a_0^2 b_0\, G=0$, so that

$$G=(b_0^8-1)\sum_{k=0}^{2} k_i a_0^i + c_3(a_0^8-1),$$

where the k_i are functions of a_1; c_3 a function of a_1, b_0. By (96_3), $b_0^2 D_{13}'$ is a multiple of a_0. Hence $b_0 c_3=0$, $c_3=k_3(b_0^8-1)$. The total coefficient of $a_0^8 b_0^8$ in D_{13} must vanish; by the terms free of b_1, $k_3=0$; by the terms in b_1, each $d_{jl}=0$. By $(96_2')$, $a_0 D_{13}$ and hence each k_i is a multiple of a_1. By (96_2), $a_1 D_{13}$ and hence $a_1 G$ is a multiple of a_0, whence $a_1 k_0=0$, $k_0=0$. By subtracting from ϕ constant multiples of $a_1^i H_a I_b$ $(i=0, 1, 2)$, we may delete from D_{13} the terms

$$a_0 a_1^r (b_0^8-1)(b_1^8-1) \qquad (r=1, 2, 3).$$

Then $k_1=0$ in G, so that

$$D_{32}=b_0 A \sum_{j=1}^{3} b_1^j \sigma_j, \quad D_{13}=(b_0^8-1)(b_1^8-1)\, k_2 a_0^2, \quad k_2=\sum_{j=1}^{3} g_j a_1^j.$$

By (95_1) and $(95_1')$, $D_{23}=(b_0^8-1)(b_1^8-1)\, F$, where F is a function of a_0 and a_1, free of a_0^8. By $(96_2')$, $a_0^8 k_2 = a_1^2 F$. Hence $k_2=0$, $D_{13}=0$. By (96_2), $a_0 F=0$, $F=0$, $D_{23}=0$. By (95_2), $(95_2')$, $D_{31}=A K$, where, as above, $A=(a_0^8-1)(a_1^8-1)$, and K is free of b_0^8. By $(96_1')$, $b_0 K = b_0 \Sigma\, b_1^{j+2} \sigma_j$, so that $K=\Sigma\, b_1^{j+2}\sigma_j$. Then (96_1) gives

$$\Sigma\, b_1^j \sigma_j = b_0^8 \Sigma\, b_1^j \sigma_j, \qquad \sigma_j = 0 \ (j=1, 2, 3).$$

We have now proved (99) and hence by (56) also (100):

$$D_{33}=D_{32}=D_{23}=D_{31}=D_{13}=0. \tag{99}$$

No term of ϕ *has a factor* $a_0^8 b_0^8$, $a_0^8 b_0^8$, $a_0^2 b_0^8$, $a_0^8 b_0$, $a_0 b_0^8$. (100)

In (94) the coefficients of a_2^2, b_2^2, a_2, b_2 now give

$$a_1 D_{30} + b_1 D_{21} = b_0^2 D_{22}, \qquad b_1 D_{03} + a_1 D_{12} = a_0^2 D_{22}, \tag{101}$$

$$a_0^2 D_{30} + b_0^2 D_{12} = b_1 D_{11}, \qquad b_0^2 D_{03} + a_0^2 D_{21} = a_1 D_{11}. \tag{102}$$

The relations derived by applying $(a_0 a_1)(b_0 b_1)$ are designated (101′), (102′).

Applying the result (100) to (101_1), we see that D_{22} does not contain a_0^3, $a_0^2 b_0$, $a_0 b_0$; nor $a_0 b_0^3$, b_0^3 by (101_2). Applying (102′) similarly, we get

$$D_{22} = a_0^2 b_0^2 d_1 + a_0^3 d_2 + a_0 d_3 + b_0^3 d_4 + b_0 d_5 + d_6 ,$$
$$D_{11} = a_0 b_0 d_7 + a_0^3 d_8 + a_0 d_9 + b_0^3 d_{10} + b_0 d_{11} + d_{12},$$

in which the d_i (and the e_i below) are functions of a_1, b_1.

By (102), D_{30} must be free of $a_0 b_0$ and a_0, since neither $a_0^3 b_0$ nor a_0^3 occur in $b_0^2 D_{12}$ or $b_1 D_{11}$. In this manner (102) and (101′) show that D_{30} is free of $a_0 b_0$, a_0, $a_0^2 b_0^2$, a_0^2; D_{03} free of $a_0 b_0$, b_0, $a_0^2 b_0^2$, b_0^2; D_{12} free of $a_0 b_0$, b_0, $a_0^2 b_0^2$, a_0^2; D_{21} free of $a_0 b_0$, a_0, $a_0^2 b_0^2$, b_0^2.

We shall simplify ϕ by subtracting constant multiples of certain invariants satisfying (99). By (57) and (61), the coefficients of $a_2^3 b_0^3$ in R^3 and Z_1 are obviously $1 + a_1^3 b_1^3$, $a_1^2 b_1$. Multiplying the former by 1, a_1^k, b_1^k $(k = 1, 2, 3)$, and the latter by $a_1^i b_1^j$ $(i, j = 0, 1, 2)$, we obtain 16 linearly independent combinations of $a_1^r a_1^s$ $(r, s = 0, 1, 2, 3)$. Hence we may assume that D_{30} is free* of b_0^3. Employing $b_1^r I_a$, we may assume that the coefficient of $a_0^3 a_0$ is a multiple of a_1. In $H_a R$, the coefficient of a_2^3 is $a_0^3 a_1^2 b_1^2 + a_0^3 a_1^3 b_0 b_1 + a_0 a_1 b_0^3$; that in $H_a R^2$, the square of the latter; that in $(a_1^3 - 1) Z_1$ is $a_0^3 b_0 (a_1^3 - 1)(b_1^3 - 1)$. Employing $a_1^i H_a R^2$, $a_1^i b_1^j H_a R$, $(a_1^3 - 1) Z_1$ $(i, j = 0, 1, 2)$, we may assume that the coefficient of $a_0^3 b_0$ in D_{30} is $\sum_{i=1}^{3} c_i b_1^i$. The coefficients of a_2^3 in $a_1^i(b_1^3 - 1) H_a R$ and $Z_1^2 + a_1^2 b_1 Z_1$ are

$$a_0 b_0^2 a_1^{i+1} (b_1^3 - 1), \qquad a_0 b_0^3 (a_1^3 - 1)(b_1^3 - 1).$$

Hence we may take the coefficient of $a_0 b_0^2 b_1^3$ to be zero.

We next subtract invariants satisfying (99) and lacking a_2^3. Employing $a_1^r I_b$ $(r = 0, 1, 2, 3)$, $a_1^i b_1^j H_b R$ $(i, j = 0, 1, 2)$, we may reduce the coefficient of b_0^3 in D_{03} to $\sum_{i=1}^{3} k_i b_1^i$. The coefficients of $(a_1^3 - 1) H_b R$ and $H_b R^2 + a_1^2 b_1^2 H_b R$ are $a_0^2 b_0 b_1 (a_1^3 - 1)$, $a_0 b_0^2 b_1^2 (a_1^3 - 1)$. Multiplying these by $b_1^i (i = 0, 1, 2)$, we may assume that in D_{03} the coefficients of $a_0^2 b_0 a_1^3$ and $a_0 b_0^2 a_1^3$ are constants.

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* The invariants used later lack $a_2^3 b_0^3$. At each stage the invariants used lack all terms previously deleted in ϕ.

Without introducing a_2^8 or b_2^8, we subtract constant multiples of $a_1^i b_1^j H_a H_b$ $(i, j = 0, 1, 2)$ and eliminate the terms $a_1^r b_1^s$ $(r, s = 1, 2, 3)$ multiplying $a_0^8 b_0^8$ in D_{22}. Hence

$$D_{30} = a_0^8 e_1 + a_0^8 b_0 \sum_{i=1}^{8} c_i b_1^i + a_0 b_0^8 e_2 + b_0^8 e_3 + b_0 e_4 + e_5,$$
$$D_{03} = a_0^8 e_6 + b_0^8 \sum_{i=1}^{8} k_i b_1^i + a_0^8 b_0 e_7 + a_0 b_0^8 e_8 + a_0^8 e_9 + a_0 e_{10} + e_{11},$$

where e_1 is a multiple of a_1, e_2 lacks b_1^8, while the coefficients of a_1^8 in e_7 and e_8 are constants, and $d_1 = \sum_{i=0}^{8} r_i a_1^i + \sum_{i=1}^{8} s_i b_1^i$. From (101′),

$$D_{12} = a_0^8 b_0 e_6 + a_0 b_0^8 (e_7 + a_1^8 d_1) + a_0 b_0 e_9 + a_0 a_1^8 d_2 + b_0^8 e_8 + b_0 e_{10} + a_1^8 d_3 + (a_0^8 - 1) e_{12},$$
$$D_{21} = a_0^8 \sum_{i=1}^{8} c_i b_1^i + a_0^8 b_0 (e_2 + b_1^8 d_1) + a_0 b_0 e_3 + a_0 e_4 + b_0 b_1^8 d_4 + b_1^8 d_5 + (b_0^8 - 1) e_{13},$$
$$a_1 d_4 = a_1 d_3 = b_1 d_2 = b_1 d_6 = 0, \quad e_1 + e_5 = b_1^8 d_3, \quad e_{11} + a_1^8 d_5 = \sum_{i=1}^{8} k_i b_1^i.$$

As shown above, D_{12} is free of $a_0 b_0$, b_0; D_{21} free of $a_0 b_0$, a_0. Hence e_3, e_4, e_9, e_{10} are zero. In relation (101_1) the coefficients of a_0^8 and $a_0^8 b_0$ give

$$a_1 e_1 = \sum_{i=1}^{8} c_i b_1^{i+1}, \quad b_1 e_2 = (b_1^8 + 1) d_1 + a_1 \Sigma c_i b_1^i.$$

By the first, each $c_i = 0$, $a_1 e_1 = 0$, $e_1 = 0$, since e_1 is a multiple of a_1. By the second and the above properties of e_2, d_1, we get $e_2 = 0$, $d_1 = 0$. The coefficients of b_0^8 and $a_0 b_0^8$ in (101_2) now give

$$a_1 e_8 = \sum_{i=1}^{8} k_i b_1^{i+1}, \quad a_1 e_7 = b_1 e_8.$$

In e_7 and e_8 the coefficients of a_1^8 are constants. Hence $k_i = 0$, $e_8 = 0$, $e_7 = c(a_1^8 - 1)$, c a constant. By the coefficient of $a_0^8 b_0^8$ in (102_1), $e_6 = 0$. The further conditions from (101) are now

$$d_2 = d_3 = d_4 = d_6 = 0, \quad d_5 = b_1 e_{13} = b_1 e_7, \quad a_1 e_5 = a_1 e_{12} = b_1 e_{11} = 0.$$

The earlier conditions now give $e_5 = 0$, $e_{11} = b_1 a_1^8 e_7 = 0$. Hence

$$D_{22} = b_0 b_1 e_7, \quad D_{30} = 0, \quad D_{03} = a_0^8 b_0 e_7, \quad D_{12} = a_0 b_0^8 e_7 + (a_0^8 - 1) e_{12},$$
$$D_{21} = (b_0^8 - 1) e_{13} + b_1^8 e_7, \quad e_7 = c(a_1^8 - 1), \quad b_1 e_{13} = b_1 e_7, \quad a_1 e_{12} = 0.$$

Then (102_1) gives $e_{12} = 0$, e_7 a multiple of b_1, whence $e_7 = 0$. Then (102_2) gives $e_{13} = 0$. Hence $b_0 D_{11} = a_0 D_{11} = 0$ by (102′). But no term has a factor $a_0^8 b_0^8$. Hence

$$D_{22} = D_{11} = D_{30} = D_{03} = D_{12} = D_{21} = 0, \tag{103}$$

No term of ϕ *has a factor* $a_0^8 b_0^8$, $a_0 b_0$, a_0^8, b_0^8, $a_0 b_0^8$, $a_0^8 b_0$. (104)

In view of (99), (100), (103), (104), we have

$$\phi = D_{00} + a_2 D_{10} + a_2^2 D_{20} + b_2 D_{01} + b_2^2 D_{02}, \tag{105}$$

where each D_{ij} is a linear function of a_0, a_0^2, b_0, b_0^2, with coefficients involving a_1, b_1. Subtracting $a_1^r b_1^s R\ (r, s \leqq 3)$ from ϕ, we may assume that D_{20} is free of b_0^2. In $a_1^i b_1^r H_a$, $b_1^{i+1}(R^2 + a_1^2 b_1^2 R)$, for $i = 0, 1, 2;\ r = 0, \ldots, 3$, the coefficients of $a_2^2 a_0^2$ are $a_1^k b_1^r\ (k = 1, 2, 3)$, b_1^{i+2}; hence in D_{20} the coefficient of a_0^2 may be made a constant l. Thus

$$D_{20} = l a_0^2 + C_1 a_0 + C_2 b_0 + C_3 \quad (C\text{'s functions of } a_1, b_1).$$

Next, H_b and $a_1^2 R^2 + a_1 b_1^2 R + b_1 H_a$ are free of a_2^2 and have $b_0^2 b_1^2$ and $b_0^2 a_1^3$ as the coefficients of b_2^2. Multiplying the former by $b_1^i a_1^r$ and the latter by $a_1^i\ (i \leqq 2,\ r \leqq 3)$, we may assume that the coefficient of b_0^2 in D_{02} is a constant λ. Thus

$$D_{02} = \lambda b_0^2 + C_4 a_0^2 + C_5 a_0 + C_6 b_0 + C_7 \quad (\lambda \text{ constant}),$$

$$D_{10} = a_0^2 C_8 + a_0 C_9 + b_0^2 C_{10} + b_0 C_{11} + C_{12}, \quad D_{01} = a_0^2 C_{13} + a_0 C_{14} + b_0^2 C_{15} + b_0 C_{16} + C_{17}.$$

For ϕ given by (105), the terms free of a_2, b_2 in (94) give

$$a_1 D_{10} + b_1 D_{01} + a_0^2 D_{20} + b_0^2 D_{02} = 0. \tag{106}$$

From this and the relation derived by $(a_0 a_1)(b_0 b_1)$, we get

$$C_i = 0\ (i = 1, \ldots, 8, 10, 12, 13, 15, 17), \quad C_9 = l a_1^2, \quad C_{11} = C_{14}, \quad C_{16} = \lambda b_1^2,$$
$$a_1 C_9 + b_1 C_{14} = l, \quad a_1 C_{11} + b_1 C_{16} = \lambda.$$

By the last two, $l = \lambda = 0$. Then $b_1 C_{11} = a_1 C_{11} = 0$,

$$C_{11} = C_{14} = c\pi, \quad \pi = (a_1^3 - 1)(b_1^3 - 1).$$

From ϕ we subtract c times the invariant

$$\pi R^2 = \pi(a_2 b_0 + b_2 a_0) = (a_1^3 + b_1^3 + 1) R^2 + a_1^2 b_1^2 R + a_1 b_1 (H_a + H_b),$$

and have every $C_i = 0$. Then $\phi = D_{00}$ is free of a_2, b_2 and hence of a_0, b_0 by (56), so that ϕ is reduced to zero by subtracting multiples of $a_1^r b_1^s\ (r, s \leqq 3)$.

The invariants which we have subtracted from ϕ are seen by inspection to be linearly equivalent to the invariants (82).

THEOREM.* *Every invariant of a pair of quadratic forms in the $GF[2^2]$*

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* In an earlier proof, I first determined the linearly independent invariants of weight $\equiv 1 \pmod 3$; then those of weight $\equiv 2$ by squaring the preceding; finally, those of weight $\equiv 0$ by noting that if J denotes the aggregate of the terms free of a_1 and b_1 in ϕ, then $\phi = \pi J + K$, where $K = (a_1^3 b_1^3 + a_1^3 + b_1^3) I$ is found by multiplying the invariants of weight 1 by a_1^2 and b_1^2, in turn. Those of type πJ contain only D_{33}, D_{30}, D_{03}, D_{21}, D_{12}, D_{00} and are found very easily.

is an integral function of the 8 *independent invariants* (81); *indeed, a linear combination of the* 144 *linearly independent invariants* (82), *for* $n=2$.

34. For the $GF[2^n]$, the general polynomial

$$\phi = \Sigma D_{rs} a_2^r b_2^s \ (r, s = 0, 1, \ldots, m = 2^n - 1) \tag{107}$$

receives under the transformation (55) an increment in which the coefficient of $a_2^\rho b_2^\sigma$ is

$$\sum_{\substack{i=0,\ldots,m-\rho \\ j=0,\ldots,m-\sigma}}' P_{ij} C_i^{\rho+i} C_j^{\sigma+j} D_{\rho+i\,\sigma+j}, \quad P_{ij} \equiv (t a_1 + t^2 a_0)^i (t b_1 + t^2 b_0)^j, \tag{108}$$

the C's being binomial coefficients and the accent denotes that i and j are not both zero. Let $T_{k\rho\sigma}$ denote the coefficient of t^k in (108), π_{kij} that of t^k in P_{ij}, after the exponents have been reduced by means of $t^{m+1} = t$. Such a reduction occurs only for $i + j \geqq 2^{n-1}$. For $n > 1$, we have*

$$\begin{aligned} T_{1\rho\sigma} = (\rho+1) a_1 D_{\rho+1\,\sigma} + (\sigma+1) b_1 D_{\rho\,\sigma+1} + \Sigma \pi_{1ij} C_i^{\rho+i} C_j^{\sigma+j} D_{\rho+i\,\sigma+j} \\ (i = 0, \ldots, m-\rho;\ j = 0, \ldots, m-\sigma;\ i + j \geqq 2^{n-1}), \end{aligned} \tag{109}$$

the final sum being absent if $\rho + \sigma > 2m - 2^{n-1}$;

$$\begin{aligned} T_{2\rho\sigma} = (\rho+1) a_0 D_{\rho+1\,\sigma} + (\sigma+1) b_0 D_{\rho\,\sigma+1} + a_1^2 C_2^{\rho+2} D_{\rho+2\,\sigma} + b_1^2 C_2^{\sigma+2} D_{\rho\,\sigma+2} \\ + a_1 b_1 (\rho+1)(\sigma+1) D_{\rho+1\,\sigma+1} + \Sigma \pi_{2ij} C_i^{\rho+i} C_j^{\sigma+j} D_{\rho+i\,\sigma+j} \\ (i \leqq m - \rho,\ j \leqq m - \sigma,\ i + j \geqq 1 + 2^{n-1}), \end{aligned} \tag{110}$$

the final sum being absent if $\rho + \sigma > 2m - 2^{n-1} - 1$.

In this modular theory, it appears to be sufficient to require the vanishing of the coefficients $T_{k\rho\sigma}$ of t^k for $k = 1, 2, 2^2, \ldots, 2^{n-1}$ (cf. *Transactions*, l. c., pp. 210, 213, 214, etc.). For $n > 2$,

$$\begin{aligned} T_{4\rho\sigma} = a_0^2 C_2^{\rho+2} D_{\rho+2\,\sigma} + b_0^2 C_2^{\sigma+2} D_{\rho\,\sigma+2} + a_0 b_0 (\rho+1)(\sigma+1) D_{\rho+1\,\sigma+1} \\ + a_0 a_1^2 C_3^{\rho+3} D_{\rho+3\,\sigma} + a_1^2 b_0 C_2^{\rho+2} (\sigma+1) D_{\rho+2\,\sigma+1} + a_0 b_1^2 C_2^{\sigma+2} (\rho+1) D_{\rho+1\,\sigma+2} \\ + b_0 b_1^2 C_3^{\sigma+3} D_{\rho\,\sigma+3} + a_1^4 C_4^{\rho+4} D_{\rho+4\,\sigma} + a_1^3 b_1 C_3^{\rho+3} (\sigma+1) D_{\rho+3\,\sigma+1} \\ + a_1^2 b_1^2 C_2^{\rho+2} C_2^{\sigma+2} D_{\rho+2\,\sigma+2} + a_1 b_1^3 C_3^{\sigma+3} (\rho+1) D_{\rho+1\,\sigma+3} + b_1^4 C_4^{\sigma+4} D_{\rho\,\sigma+4} \\ + \Sigma \pi_{4\rho\sigma} C_i^{\rho+i} C_j^{\sigma+j} D_{\rho+i\,\sigma+j} \ (i \leqq m - \rho,\ j \leqq m - \sigma,\ i + j \geqq 2 + 2^{n-1}). \end{aligned} \tag{111}$$

If we write ρ and i to the scale of base 2, $C_i^{\rho+i}$ is odd if each coordinate of i is less than or equal to the corresponding coordinate of ρ, viz., if the partition of $\rho + i$ into ρ and i takes place in the coefficients of the various powers of 2

* Here and below, terms preceding the summation signs are to be suppressed if they contain a D with subscript $> m$.

separately. In the contrary case, C_i^{l+i} is a multiple of the modulus 2. For example, since $m = 2^n - 1$, we have when $n > 1$,

$$C_2^{m+1} \equiv 0, \quad C_2^{m+2} \equiv 0, \quad C_2^{m+3} \equiv 1 \pmod{2}.$$

Hence we have, by inspection,

$$T_{1mm-1} \equiv b_1 D_{mm}, \quad T_{1m-1m} \equiv a_1 D_{mm}, \quad T_{2mm-1} \equiv b_0 D_{mm}, \quad T_{2m-1m} \equiv a_0 D_{mm}.$$

For an invariant ϕ, these must vanish. Thus

$$D_{mm} = d\,(a_0^m - 1)(b_0^m - 1)(a_1^m - 1)(b_1^m - 1).$$

Hence $\phi - d\,I_a I_b$ has $D_{mm} = 0$. For $n \geqq 3$, various D's are now necessarily zero, so that the above T's simplify materially. In fact, the special relations discussed at any stage may be chosen so that the coefficients of the D's involve a_i and b_i only in the combinations $a_0^{2^i}$, $b_0^{2^i}$, $a_1^{2^i}$, $b_1^{2^i}$. Thus the effective parts* of the conditions (108), when used in a convenient sequence, are given by $i = 0$ or $j = 0$ and so may be determined by inspection.

35. Let next $n = 3$. As in § 34, we may set $D_{77} = 0$. By (56) no term of ϕ has the factor $a_0^7 b_0^7$. Thus $a_0 D = b_0 D = 0$ imply $D = 0$. In T_{171}, $i = 0$, $j = 4, 5, 6$; but $j = 6$ leads to D_{77}, $j = 5$ gives $C_5^{1+5} \equiv 0$, while $j = 4$ gives $b_0^4 D_{75} = 0$. Similarly, T_{185} gives $a_0^4 D_{75} = 0$. Permuting the a's and b's, $T_{117} = a_0^4 D_{57}$, $T_{158} = b_0^4 D_{57}$. Hence $D_{75} = D_{57} = 0$. For $D_{77} = 0$,

$$T_{272} = b_0 D_{78}, \quad T_{268} = a_0 D_{78}, \quad T_{227} = a_0 D_{87}, \quad T_{286} = b_0 D_{87},$$

whence $D_{78} = D_{87} = 0$. In view of the latter,

$$T_{172} = b_0^4 D_{76}, \quad T_{168} = b_0^4 D_{67}, \quad T_{127} = a_0^4 D_{67}, \quad T_{186} = a_0^4 D_{76},$$

whence $D_{76} = D_{67} = 0$. Next,

$$T_{151} = b_0^4 D_{55}, \quad T_{288} = b_0 D_{88}, \quad T_{464} = b_0^2 D_{66};$$

thus $T_{115} = a_0^4 D_{55}$, etc., so that the three D's vanish. Hence

$$D_{77} = D_{76} = D_{75} = D_{78} = D_{67} = D_{66} = D_{57} = D_{55} = D_{87} = D_{88} = 0. \quad (112)$$

The remaining 54 D's have non-vanishing values in R^7 or $H_a R^7$. By (56),

No term of ϕ *contains* $a_0^7 b_0^i$, $a_0^i b_0^7$ $(i = 7, 6, 5, 3)$, $a_0^6 b_0^6$, $a_0^5 b_0^5$, $a_0^3 b_0^3$. (113)

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* In the final sums in (109), .., certain of the π's vanish for $n > 2$. For example, if $n = 3$, $\pi_{1ij} = 0$ for ij or $ji = 50, 41, 54, 64$; $\pi_{2ij} = 0$ for $i + j = 6$ $(i \neq 3)$; $\pi_{4ij} = 0$ for $ij = 60, 42, 24, 06, 62, 44, 26$. But at the stage at which the relation is used the coefficient of such a factor π is zero, so that there is no gain in employing that fact that certain of the π vanish.

18

After deleting the D's in (112), $T_{\kappa\rho\sigma}$ for $\kappa = 1, 2, 4$, $\rho, \sigma = 70, 61, 52, 43, 64, 62, 54, 51, 32, 31$ give binomial relations* involving only the following twelve D_{ij}:

$$D_{74},\ D_{72},\ D_{71},\ D_{65},\ D_{63},\ D_{56},\ D_{53},\ D_{47},\ D_{36},\ D_{35},\ D_{27},\ D_{17}. \tag{114}$$

$$b_1 D_{71} = b_0^4 D_{74},\quad a_1 D_{71} = b_0^4 D_{65},\quad b_1 D_{53} = b_0^4 D_{56},\quad a_1 D_{53} = b_0^4 D_{47}, \tag{115}$$

$$a_1 D_{74} = b_1 D_{65},\quad a_1 D_{72} = b_1 D_{63},\quad a_0^4 D_{72} = b_0^4 D_{36},\quad a_0^4 D_{71} = b_0^4 D_{35}; \tag{116}$$

$$b_0 D_{71} = b_1^2 D_{72},\quad a_0 D_{71} = b_1^2 D_{63},\quad b_0 D_{53} = a_1^2 D_{72},\quad a_0 D_{53} = a_1^2 D_{63}, \tag{117}$$

$$a_0 D_{74} = b_0 D_{65},\quad a_0 D_{72} = b_0 D_{63},\quad a_1^2 D_{74} = b_1^2 D_{56},\quad a_1^2 D_{71} = b_1^2 D_{53}; \tag{118}$$

$$b_0^2 D_{72} = b_1^4 D_{74},\quad b_0^2 D_{63} = b_1^4 D_{65},\quad a_0^2 D_{72} = b_1^4 D_{56},\quad a_0^2 D_{63} = b_1^4 D_{47}, \tag{119}$$

$$a_0^2 D_{74} = b_0^2 D_{56},\quad a_0^2 D_{71} = b_0^2 D_{53},\quad a_1^4 D_{72} = b_1^4 D_{36},\quad a_1^4 D_{71} = b_1^4 D_{35}. \tag{120}$$

The binomial relations $T_{\kappa\sigma\rho}$, with the same ρ, σ, may be derived by interchanging the a's and b's (and hence permuting the subscripts of D_{ij}); they will be designated (115′).

By $b_0(116_1) + b_1(118_1)$ and $b_0^3(115_1) + b_0^2 b_1(117_1) + b_1^3(119_1)$,

$$(a_1 b_0 + a_0 b_1) D_{74} = 0, \qquad (b_0^7 + b_1^7) D_{74} = 0.$$

The discussion of these is similar to that in § 33. By the second and $(115_4')$,

$$D_{74} = \Sigma \delta_{ij} b_0^i b_1^j \qquad (i, j = 1, \ldots, 7). \tag{121}$$

Then the first gives (98) for $i, j \leq 7$. But, by (113), δ_{7j} is free of a_0^7, a_0^6, a_0^5, a_0^3. Then (98), for $i = 7$, $i = 1$, shows that δ_{7j} is a multiple of $a_1 a_0$:

$$\delta_{7j} = a_0^4 \sum_{k=1}^{7} c_{jk} a_1^k + a_0^2 \sum_{k=1}^{7} d_{jk} a_1^k + a_0 \sum_{k=1}^{7} e_{jk} a_1^k \qquad (j = 1, \ldots, 7),$$

the c, d, e being constants whose subscripts may be reduced modulo 7. Then (98), for $i = 7$, gives

$$\delta_{6j} = a_0^5 C_j + a_0^3 D_j + a_0^2 E_j + (a_1^7 - 1)\Gamma_j, \qquad C_j = \sum_{k=1}^{7} c_{j-1k} a_1^{k-1}, \ldots.$$

We may introduce a term from Γ_j into C_j and set

$$C_j = \sum_{k=2}^{8} c_{j-1k} a_1^{k-1} = \sum_{l=1}^{7} c_{j-1\,l+1} a_1^l \qquad (k = l + 1).$$

A similar modification may be made in D_j, E_j. By (98), for $i = 6$, $a_0 \delta_{6j-1}$ and hence $a_0 \Gamma_{j-1}$ is a multiple of a_1; thus $a_0 \Gamma_{j-1} = 0$. By (113), δ_{6j} lacks a_0^7. Hence $\Gamma_{j-1} = 0$ for every j, so that

$$\delta_{6j} = a_0^5 \sum_{l=1}^{7} c_{j-1\,l+1} a_1^l + a_0^3 \sum_{l=1}^{7} d_{j-1\,l+1} a_1^l + a_0^2 \sum_{l=1}^{7} e_{j-1\,l+1} a_1^l.$$

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* Of these T_{464}, T_{462}, T_{154}, T_{151}, T_{232}, T_{231} give identities.

By (98), for $i=6$, $\delta_{5j}=a_0^6 C'_j + a_0^4 D'_j + a_0^8 E'_j + (a_1^7-1)\Gamma'_j$, where we may set

$$C'_j = \sum_{\lambda=1}^{7} c_{j-2\,\lambda+1}\, a_1^{\lambda-1} + c_{j-28}(a_1^7-1) = \sum_{\lambda=2}^{8} c_{j-2\,\lambda+1}\, a_1^{\lambda-1} = \sum_{l=1}^{7} c_{j-2\,l+2}\, a_1^l,$$

and similarly for D'_j, E'_j. By (98), for $i=5$, $a_0\,\delta_{5j-1}$ and hence $a_0\,\Gamma'_{j-1}$ is a multiple of a_1. But Γ' lacks a_0^7 by (113). Hence $\Gamma'=0$,

$$\delta_{5j} = a_0^6 \sum_{l=1}^{7} c_{j-2\,l+2}\, a_1^l + a_0^4 \sum_{l=1}^{7} d_{j-2\,l+2}\, a_1^l + a_0^8 \sum_{l=1}^{7} e_{j-2\,l+2}\, a_1^l,$$

$$\delta_{4j} = a_0^7 \sum_{l=1}^{7} c_{j-3\,l+3}\, a_1^l + a_0^5 \sum_{l=1}^{7} d_{j-3\,l+3}\, a_1^l + a_0^4 \sum_{l=1}^{7} e_{j-3\,l+3}\, a_1^l + (a_1^7-1)\,\varepsilon_j.$$

By (98), for $i=4$, $a_0\,\delta_{4j-1}$ and hence $a_0\,\varepsilon_{j-1}$ is a multiple of a_1. Thus

$$\varepsilon_j = e_j\,(a_0^7-1),\ e_j \text{ a constant.}$$

Now the coefficients of $a_0^7 a_2^7 b_2^4$ in $H_b\, I_a$ and $H_a H_b R^3$, which satisfy (112), are

$$(a_1^7-1)\, b_0^4 b_1^6, \qquad a_1^6 b_0^4 b_1^6.$$

Multiplying the former by b_1^i and the latter by $a_1^i b_1^j$ $(i, j = 0, \ldots, 6)$, and subtracting constant multiples of the products from ϕ, we may delete the terms

$$a_0^7 b_0^4 b_1^r, \qquad a_0^7 b_0^4 a_1^r b_1^s \qquad (r, s = 1, \ldots, 7),$$

from D_{74}. Then δ_{4r} lacks a_0^7. Hence the total coefficient of a_0^7 in the above expression for δ_{4j} must vanish. Thus

$$c_{jk}=0, \qquad \varepsilon_j = 0 \qquad (j, k = 1, \ldots, 7).$$

Since δ_{3j} lacks a_0^7, by (113), the earlier argument gives

$$\delta_{3j} = a_0^6 \sum_{l=1}^{7} d_{j-4\,l+4}\, a_1^l + a_0^5 \sum_{l=1}^{7} e_{j-4\,l+4}\, a_1^l,$$

$$\delta_{2j} = a_0^7 \sum_{l=1}^{7} d_{j-5\,l+5}\, a_1^l + a_0^6 \sum_{l=1}^{7} e_{j-5\,l+5}\, a_1^l + s_j A,$$

$$\delta_{1j} = a_0 \sum_{l=1}^{7} d_{j-6\,l+6}\, a_1^l + a_0^7 \sum_{l=1}^{7} e_{j-6\,l+6}\, a_1^l + t_j A,$$

where $A=(a_0^7-1)(a_1^7-1)$, s and t constants. Since the subscripts of d and e are taken modulo 7, (121) may not be written in the form

$$D_{74} = \sum_{i,j}^{1,\ldots,7} \{ b_0^i b_1^j (a_0^{9-i} \sum_{l=1}^{7} d_{j+i\,l-i}\, a_1^l + a_0^{8-i} \sum_{l=1}^{7} e_{j+i\,l-i}\, a_1^l) + b_1^j B_j A \},$$

where $B_j = b_0^3 s_j + b_0 t_j$. Then $(115_4')$ gives

$$D_{35} = \sum_{i,j}^{1,\ldots,7} b_0^i b_1^{j-1} \left(a_0^{13-i} \sum_{l=1}^{7} d_{j+i\, l-i}\, a_1^l + a_0^{12-i} \sum_{l=1}^{7} e_{j+i\, l-i}\, a_1^l\right) + (b_1^7 - 1)\, G,$$

where G is a function of a_0, a_1, b_0. Now

$$a_1^4 b_1^3 (115_3') + a_0^4 b_0^2 (117_3') + a_0^4 b_1^2 (119_3') : (a_0^5 b_0^3 + a_1^5 b_1^3)\, D_{35} = 0.$$

After a simple change of summation indices, this reduces to

$$a_0^5 b_0^3\, G = 0.$$

For $G = \Sigma c_i a_0^i$, the latter gives $b_0 c_i = 0$ $(i = 1, \ldots, 6)$, $b_0 (c_0 + c_7) = 0$,

$$G = (b_0^7 - 1) \sum_{i=0}^{6} k_i a_0^i + c_7 (a_0^7 - 1),$$

where the k_i are functions of a_1; c_7 a function of a_1, b_0, free of b_0^7. By $(120_2')$, $a_0^2 D_{35}$ is a multiple of b_0. Hence $a_0^2 \sum_{i=0}^{6} k_i a_0^i = 0$, $k_i = 0$. By (116_4), $b_0^4 D_{35}$ and hence $b_0^4 c_7$ is a multiple of a_0. Thus $c_7 = 0$, $G = 0$. By (113), D_{35} lacks $a_0^7 b_0^6$, $a_0^6 b_0^6$. Hence every d and e vanish. Thus

$$D_{35} = 0, \quad D_{74} = \beta A, \quad \beta = \sum_{j=1}^{7} b_1^j (b_0^3 s_j + b_0 t_j), \quad A = (a_0^7 - 1)(a_1^7 - 1).$$

By $(115_3')$, (118_1), $a_0 D_{65} = b_0 D_{65} = 0$, whence $D_{65} = 0$ by (113). By $(118_4')$, $b_1 D_{17} = 0$. Then by $(115_2')$, (120_1), $a_0 D_{56} = b_0 D_{56} = 0$, $D_{56} = 0$. Thus $(119_2')$, $(119_4')$, $D_{36} = 0$. By $(116_1')$, $b_1 D_{47} = 0$. Then by (119_2) and (119_4), $D_{63} = 0$. By (116_4), $a_0 D_{71} = 0$. Then (120_2) and (117_4), $D_{53} = 0$. By (116_2), (118_3), $D_{72} = AQ$, where by (113), Q lacks b_0^7, b_0^6, b_0^5, b_0^3. Then (119_1) requires that Q be independent of b_0. By (117_1), $b_1^2 D_{72} = b_0 D_{71} = 0$. Hence $Q = c\,(b_1^7 - 1)$. As above $a_0 D_{71} = 0$, whence $D_{71} = 0$. Then $D_{74} = 0$ by (115_1), $D_{73} = 0$ by (119_1). By $(116')$, $(115_2')$, (115_4), D_{47}, D_{27}, D_{17} are multiples of $B = (b_0^7 - 1)(b_1^7 - 1)$. Then by (113) and $(115_1')$, $(117_1')$, $(119_1')$, we get $D_{47} = a_0^4 r L$, $D_{27} = a_0^2 r a_1^4 L$, $D_{17} = a_0 r a_1^6 L$, where r is a function of a_1 such that $r(a_1^7 - 1) = 0$, and hence a multiple of a_1. Hence after subtracting constant multiples of $a_1^4 H_a I_b$, we have $r = 0$. Hence all the D_{ij} in (114) now vanish. Then by (56), ϕ has no

$$a_0^7 b_0^i,\ a_0^i b_0^7\ (i = 4, 2, 1),\ a_0^6 b_0^5,\ a_0^5 b_0^6,\ a_0^6 b_0^3,\ a_0^3 b_0^6,\ a_0^5 b_0^3,\ a_0^3 b_0^5. \tag{122}$$

Since the 22 D's in (112) and (114) vanish, T_{kll} $(k, l = 1, 2, 4)$ give

$$a_0^4 D_{51} = b_0^4 D_{15}, \quad a_1 D_{64} = b_1 D_{45}, \quad a_1 D_{32} + b_1 D_{23} = a_0^4 D_{62} + b_0^4 D_{26}, \tag{123}$$

$$a_0 D_{32} = b_0 D_{23}, \quad a_1^2 D_{31} = b_1^2 D_{13}, \quad a_1^2 D_{64} + b_1^2 D_{46} = a_0 D_{54} + b_0 D_{45}, \tag{124}$$

$$a_0^2 D_{64} = b_0^2 D_{46}, \quad a_1^4 D_{62} = b_1^4 D_{26}; \quad a_1^4 D_{51} + b_1^4 D_{15} = a_0^2 D_{31} + b_0^2 D_{13}, \tag{125}$$

while T_{160}, T_{142}, T_{150}, T_{141}, T_{250}, T_{241}, T_{230}, T_{221}, T_{460}, T_{442}, T_{430}, T_{421} give

$$a_1 D_{70} + b_1 D_{61} = b_0^4 D_{64},\ a_1 D_{52} + b_1 D_{43} = b_0^4 D_{46},\ b_1 D_{51} = b_0^4 D_{54},\ a_1 D_{51} = b_0^4 D_{45}, \tag{126}$$

$$a_1^2 D_{70} + b_1^2 D_{52} = b_0 D_{51},\ a_1^2 D_{61} + b_1^2 D_{43} = a_0 D_{51},\ b_1^2 D_{32} = b_0 D_{31},\ b_1^2 D_{23} = a_0 D_{31}, \tag{127}$$

$$a_1^4 D_{70} + b_1^4 D_{34} = b_0^2 D_{32},\ a_1^4 D_{61} + b_1^4 D_{25} = b_0^2 D_{23},\ b_1^4 D_{64} = b_0^2 D_{62},\ b_1^4 D_{46} = a_0^2 D_{62}. \tag{128}$$

The relations derived from the latter by interchanging the a's and b's, D_{ij} and D_{ji}, will be designated (126′), etc. In any D_{ij} certain factors are lacking by (113), (122). Then by (126), (126′), D_{64} and D_{46} lack also the factors

$$a_0^i b_0^j,\quad ij \text{ or } ji = 70,\ 62,\ 61,\ 52,\ 51,\ 43,\ 32,\ 31.$$

Applying also (125_1), we have the first two of the relations

$$D_{64} = b_0^2 D,\quad D_{46} = a_0^2 D,\quad D_{62} = b_1^4 D,\quad D_{26} = a_1^4 D, \tag{129}$$

$$D = d_1 a_0^6 b_0^2 + d_2 a_0^4 b_0^4 + d_3 a_0^4 + d_4 a_0^2 b_0^2 + d_5 a_0^2 + d_6 a_0^2 b_0^6 + d_7 a_0 + d_8 b_0^4 + d_9 b_0^2 + d_{10} b_0 + d_{11},$$

the d_i being functions of a_1, b_1. By (128_3), $(128'_4)$,

$$D_{62} = b_1^4 D + (b_0^7 - 1)\, m,\quad D_{26} = a_1^4 D + (a_0^7 - 1)\, l.$$

By $(128'_3)$ and (128_4), $a_0 l = a_0 m = 0$, $l = m = 0$, whence the final relations (129).

By (127), (127′), D_{51} and D_{15} lack also the factors

$$a_0^i b_0^j,\quad ij \text{ or } ji = 70,\ 64,\ 62,\ 61,\ 54,\ 52,\ 43,\ 32.$$

Applying also (123_1), we have the first two of the relations

$$D_{51} = b_0^4 E,\quad D_{15} = a_0^4 E,\quad D_{54} = b_1 E,\quad D_{45} = a_1 E, \tag{130}$$

$$E = e_1 a_0^5 b_0^4 + e_2 a_0 b_0 + e_3 a_0 + e_4 a_0^4 b_0^4 + e_5 a_0^4 + e_6 a_0^4 b_0^5 + e_7 a_0^2 + e_8 b_0 + e_9 b_0^4 + e_{10} b_0^2 + e_{11},$$

the literal terms being the same as in D^2. By (126_3), (126_4),

$$D_{54} = b_1 E + (b_0^7 - 1)\, \varepsilon,\quad D_{45} = a_1 E + (b_0^7 - 1)\, \varepsilon_1.$$

By $(126'_3)$, $(126'_4)$, $a_0 \varepsilon_1 = a_0 \varepsilon = 0$, $\varepsilon_1 = \varepsilon = 0$, whence $(130_{3,4})$.

Similarly, by (128), (128′), D_{32} and D_{23} lack also the factors

$$a_0^i b_0^j,\quad ij \text{ or } ji = 70,\ 64,\ 61,\ 54,\ 52,\ 51,\ 43,\ 31.$$

Using (124_1), (127_3), $(127'_4)$, $(127'_3)$, (127_4), we get

$$D_{32} = b_0 F,\quad D_{23} = a_0 F,\quad D_{31} = b_1^2 F,\quad D_{13} = a_1^2 F, \tag{131}$$

$$F = f_1 a_0^3 b_0 + f_2 a_0^3 b_0^2 + f_3 a_0^2 + f_4 a_0 b_0 + f_5 a_0 + f_6 a_0 b_0^3 + f_7 a_0^4 + f_8 b_0^2 + f_9 b_0 + f_{10} b_0^4 + f_{11},$$

the literal terms being the same as in E^2 or D^4.

In terms of $r = a_0 b_1 + a_1 b_0$, conditions (123_3), (124_3), (125_3) become

$$r F = r^4 D,\quad r E = r^2 D,\quad r^4 E = r^2 F,$$

of which the last follows from the first two. The first two are satisfied if and only if d_i, e_i, f_i ($i = 3, 4, 5, 8, 9, 11$) are constant multiples* of

$$\pi = (a_1^7 - 1)(b_1^7 - 1); \tag{132}$$

$$\left.\begin{array}{l} a_1^4 d_1 = b_1^4 d_6,\ b_1^4 d_7 = b_1 f_7 = b_1^3 e_7,\ a_1^4 d_{10} = a_1 f_{10} = a_1^3 e_{10},\ a_1 e_1 = b_1 e_6, \\ a_1 f_7 + b_1 f_1 + a_1^4 d_2 + b_1^4 d_{10} = 0,\quad a_1 f_1 + b_1 f_2 + b_1^4 d_1 = 0, \\ a_1 f_6 + b_1 f_{10} + b_1^4 d_2 + a_1^4 d_7 = 0,\quad a_1 f_2 + b_1 f_6 + a_1^4 d_6 = 0, \\ a_1^2 d_7 + b_1^2 d_1 + a_1 e_2 + b_1 e_{10} = 0,\quad a_1^2 d_1 + b_1^2 d_2 + b_1 e_1 = 0, \\ a_1^2 d_6 + b_1^2 d_{10} + b_1 e_2 + a_1 e_7 = 0,\quad a_1^2 d_2 + b_1^2 d_6 + a_1 e_6 = 0. \end{array}\right\} \tag{133}$$

In the following invariants, having the D_{ij} in (112) and (114) zero,

$$H_a R^5,\ H_b R^5,\ H_a H_b R,\ R^7,\ H_b R^3,\ H_a R^3,\ H_a H_b R^2,\ H_b R^6,\ H_a R^6,$$

the coefficients of $a_2^6 b_2^4$ are, respectively,

$$a_0^4 b_0^6 a_1^7 + a_0 b_0^2 a_1^3 b_1^4,\quad a_0^4 b_0^6 b_1^7 + b_0^3 a_1^4 b_1^3,\quad a_0^4 b_0^6 a_1^6 b_1^6,\quad a_0 b_0^2 b_1 + b_0^3 a_1,$$
$$b_0^3 b_1^6,\quad a_0 b_0^2 a_1^6,\quad a_0^6 b_0^4 a_1^6 b_1^3 + a_0^4 b_0^6 a_1 b_1 + a_0^2 b_0 a_1^3 b_1^6,$$
$$a_0^6 b_0^4 b_1^4 + a_0^2 b_0 a_1^4 b_1^7 + a_0^4 b_0^6 a_1^2 b_1^2 + b_0^3 a_1^6 b_1^5,$$
$$a_0^6 b_0^4 a_1^7 b_1^4 + a_0^2 b_0 a_1^4 + a_0^4 b_0^6 a_1^2 b_1^2 + a_0 b_0^2 a_1^5 b_1^6.$$

Subtracting the products of the first by a_1^i, the second by b_1^i, the third by $a_1^i b_1^j$ ($i, j = 0, \ldots, 6$), we may assume that the coefficient d_2 of $a_0^4 b_0^6$ in D_{64} is a constant. Subtracting $a_1^i R^7$, $a_1^r b_1^i H_b R^3$ ($i \leqq 6$, $r \leqq 7$), we make the coefficient d_{10} of b_0^3 in D_{64} a constant. Subtracting $a_1^i b_1^r H_a R^3$ and $b_1^{i+1} P$, where† $P = b_1 R^7 + a_1 b_1^2 H_b R^3$ has $a_0 b_0^2 b_1^2$ as the coefficient of $a_2^6 b_2^4$, we make the coefficient d_7 of $a_0 b_0^2$ in D_{64} a constant. The coefficients of $a_2^6 b_2^4$ in

$$S = H_a H_b R^2 + a_1^2 b_1^2 H_a H_b R,\qquad H_b R^6 + a_1^2 b_1^2 H_b R^5,$$
$$H_a R^6 + a_1^2 b_1^2 H_a R^5 + a_1 b_1 S$$

are $a_0^6 b_0^4 a_1^6 b_1^3 + a_0^2 b_0 a_1^3 b_1^6$, $a_0^6 b_0^4 b_1^4 + a_0^2 b_0 a_1^4 b_1^7$, $a_0^2 b_0 a_1^4 (b_1^7 - 1)$, respectively. Subtracting the products of S by $a_1^i b_1^j$ and the second by b_1^i ($i, j \leqq 6$), we make the coefficient d_1 of $a_0^6 b_0^4$ a function of a_1 alone. Subtracting the products of the third by a_1^i ($i \leqq 6$), we make the coefficient of $a_0^2 b_0 b_1^7$ a constant, so that in d_6 the coefficient of b_1^7 is constant. Then by (133_1),

$$d_1 = \delta_1 (a_1^7 - 1),\qquad d_6 = \delta_6 (b_1^7 - 1)\qquad (\delta_1, \delta_6 \text{ constants}).$$

* $a_1 d = b_1 d = 0$ imply that d is a constant multiple of π.

† In the last line of (82) occurs also $a_1^7 R^7$, which we may here replace by $(a_1^7 + b_1^7 + 1) R^7$ and then, in view of (68_1), by $(a_1^7 - 1)(b_1^7 - 1) R^7$. The latter is used in the next paragraph; see (140).

The d_i, other than these and the constants d_2, d_7, d_{10}, were seen to be multiples of π, defined by (132). But, by (126_1), $b_0^4 D_{64}$ has every term a multiple of a_1 or b_1. The same is true of each d_i, since a_0 and b_0 do not enter the same manner in two terms of $b_0^4 D_{64}$, by (129_1). Hence every $d_i = 0$. Then a simple discussion of conditions (133) shows that the e_i, f_i occurring in them are all multiples of π. Hence every e_i, f_i $(i \leqq 11)$ is a constant multiple of π. But by (127_1) and $(128_2')$, $a_0^2 D_{33}$ and $b_0 D_{51}$ are multiples of a_1 or b_1. As above, each $e_i = f_i = 0$. Hence

$$D_{64} = D_{62} = D_{54} = D_{51} = D_{46} = D_{45} = D_{32} = D_{31} = D_{26} = D_{23} = D_{15} = D_{13} = 0, \qquad (134)$$

No term of ϕ *has a factor* $a_0^6 b_0^4, \ldots, a_0 b_0^3$. (135)

With the vanishing D's deleted, T_{130}, T_{131}, T_{260}, T_{242}, T_{450}, T_{441} give

$$a_0^4 D_{70} = b_0^4 D_{34}, \quad a_0^4 D_{61} = b_0^4 D_{25}, \quad a_0 D_{70} = b_0 D_{61}, \qquad (136)$$
$$a_0 D_{52} = b_0 D_{43}, \quad a_0^2 D_{70} = b_0^2 D_{52}, \quad a_0^2 D_{61} = b_0^2 D_{43}. \qquad (137)$$

The possible factors $a_0^i b_0^j$ are now those in which i, j are

$$\left.\begin{array}{r} 70,\ 61,\ 60,\ 52,\ 50,\ 44,\ \ldots\ 40,\ 34,\ 30,\ 25,\ 24,\ 22,\ 21,\ 20, \\ 16,\ 14,\ 12,\ 11,\ 10,\ 07,\ \ldots,\ 00. \end{array}\right\} \qquad (138)$$

By (136_3), D_{70} cannot contain $a_0^i b_0^j$, $ij = 60, \ldots, 10, 44, 42, 24, 22, 14, 06, 04$. By (137_2), also 41, 21, 11, 01, 05 are absent; by (136_1), also 12, 03, 02. Hence

$$D_{70} = \sum_{i=1}^{7} g_i a_0^{7-i} b_0^i + g_0 (a_0^7 - 1) \qquad (g\text{'s functions of } a_1, b_1).$$

Then by (136_3),

$$D_{61} = \sum_{i=1}^{7} g_i a_0^{8-i} b_0^{i-1} + g_8 (b_0^7 - 1) \equiv \sum_{j=1}^{7} g_{j+1} a_0^{7-j} b_0^j + g_1 a_0^7 - g_8.$$

By (137_3), $a_0^2 (g_1 a_0^7 - g_8) = 0$, $g_8 = g_1$. Then (137_2) gives

$$D_{52} = \sum_{i=1}^{7} g_i a_0^{9-i} b_0^{i-2} + g_9 (b_0^7 - 1) \equiv \sum_{j=1}^{7} g_{j+2} a_0^{7-j} b_0^j + g_2 a_0^7 - g_9.$$

By (137_1), $a_0 (g_2 a_0^7 - g_9) = 0$, ... $= g_2$, and

By $(137'_1)$, $g_{11} = g_4$, and

$$D_{25} = \sum_{i=1}^{7} g_{i+4}\, a_0^{8-i}\, b_0^{i-1} + g_{12}\,(b_0^7 - 1) \equiv \sum_{j=1}^{7} g_{j+5}\, a_0^{7-j}\, b_0^j + g_5\, a_0^7 - g_{12}.$$

Then $(137'_2)$ and $(137'_3)$ give $g_{12} = g_5$ and

$$D_{16} = \sum_{i=1}^{7} g_{i+4}\, a_0^{9-i}\, b^{i-2} + g_{13}\,(b_0^7 - 1) \equiv \sum_{j=1}^{7} g_{j+6}\, a_0^{7-j}\, b_0^j + g_6\, a_0^7 - g_{13},$$

$$D_{07} = \sum_{i=1}^{7} g_{i+5}\, a_0^{9-i}\, b_0^{i-2} + g_{14}\,(b_0^7 - 1) \equiv \sum_{j=1}^{7} g_{j+7}\, a_0^{7-j}\, b_0^j + g_7\, a_0^7 - g_{14}.$$

By $(136'_3)$, $g_{13} = g_6$. Thus $g_{7+k} = g_k$ $(k = 1, \ldots, 6)$. All the conditions (136)–(137′) are now satisfied. By (126_1), $(126'_1)$, .., (128_1), $(128'_1)$,

$$a_1\, g_i = b_1\, g_{i+1}, \quad a_1^2\, g_i = b_1^2\, g_{i+2}, \quad a_1^4\, g_i = b_1^4\, g_{i+4}, \tag{139}$$

for every i making no subscript > 14. Conditions (126_2), $(126'_2)$, .., are satisfied. By (139),

$$a_1^7\, g_1 = a_1^6\, b_1\, g_2 = a_1^4\, b_1 \,.\, b_1^2\, g_4 = b_1^3 \,.\, b_1^4\, g_1, \quad (a_1^7 + b_1^7)\, g_1 = 0,$$

$$g_1 = \sum_{i,j}^{1,\ldots,7} c_{ij}\, a_1^i\, b_1^j + c_{00}\,(1 + a_1^7 + b_1^7) \quad (c\text{'s constants}).$$

In $Z_1 = V_1$ and $Z_1^2 = V_2$, given by (61), the coefficients of $a_2^7\, a_0^6\, b_0$ are $1 + a_1^7 + b_1^7$ and $a_1^6\, b_1$. Hence by subtracting constant multiples of Z_1 and $a_1^i b_1^j Z_1^2$, we may take $g_1 = 0$. Then by (139) for $i = 1$, $b_1\, g_2 = 0$, $g_2 = A\,(b_1^7 - 1)$, where A is a function of a_1 alone. By $a_1\, g_2 = b_1\, g_3$, $a_1 A$ is a multiple of b_1 and hence zero. Thus $A = l_2\,(a_1^7 - 1)$. Proceeding similarly, we find that (139) gives

$$g_i = l_i\, \pi \;(i = 2, \ldots, 7), \quad g_0 = \beta\,(a_1^7 - 1), \quad g_{14} = \alpha\,(b_1^7 - 1),$$

where π is given by (132), β a function of b_1, α a function of a_1, l_i constants. Subtracting $\beta\, I_a$, in which the coefficient of $a_2^7\,(a_0^7 - 1)$ is $\beta\,(a_1^7 - 1)$, we have $g_0 = 0$ in D_{70}. Subtracting $\alpha\, I_b$, in which the coefficient of $b_2^7\,(b_0^7 - 1)$ is $\alpha\,(b_1^7 - 1)$, we have $g_{14} = 0$ in D_{07}. In

(140)

the

multiples of these and πR^7, we have $g_i = 0$ $(i = 2, .., 7)$ in D_{70}. Hence every $g_i = 0$,

$$D_{70} = D_{61} = D_{52} = D_{43} = D_{34} = D_{25} = D_{16} = D_{07} = 0, \tag{141}$$

No term of ϕ *has a factor* $a_0^{7-i} b_0^i$ $(i = 0, .., 7)$. (142)

From the latter and (138), the possible factors $a_0^i b_0^j$ now have

$$i, j = 60, .., 00, 06, .., 01, 44, 42, 41, 24, 22, 21, 14, 12, 11; \tag{143}$$

while D_{ij} vanishes unless i, j is one of these pairs. Then T_{kl0} $(k, l = 1, 2, 4)$ give

$$a_0^4 D_{50} + b_0^4 D_{14} = b_1 D_{11},\quad a_1 D_{50} + b_1 D_{41} = b_0^4 D_{44},\quad a_1 D_{30} + b_1 D_{21} = a_0^4 D_{60} + b_0^4 D_{24}, \tag{144}$$
$$a_0 D_{30} + b_0 D_{21} = b_1^2 D_{22},\quad a_1^2 D_{30} + b_1^2 D_{12} = b_0 D_{11},\quad a_1^2 D_{60} = b_1^2 D_{42} = a_0 D_{50} + b_0 D_{41}, \tag{145}$$
$$a_0^2 D_{60} + b_0^2 D_{42} = b_1^4 D_{44},\quad a_1^4 D_{60} + b_1^4 D_{24} = b_0^2 D_{22},\quad a_1^4 D_{50} + b_1^4 D_{14} = a_0^2 D_{30} + b_0^2 D_{12}. \tag{146}$$

We subtract from ϕ constant multiples of invariants containing only terms (143). Subtracting $a_1^r b_1^s R^3$ $(r, s \leqq 7)$, we delete b_0^6 in D_{60}. The coefficients of a_2^6 in $H_a R$ and $R^5 + a_1^4 b_1^4 R^3$ are $a_0^4 b_0^2 a_1^6$, $a_0^4 b_0^2 b_1$. Subtracting the product of the first by $a_1^i b_1^r$ and the product of the second by b_1^{i+1} $(i \leqq 6, r \leqq 7)$, we make the coefficient of $a_0^4 b_0^2$ in D_{60} a constant. The coefficients of a_2^6 in $h = H_a R^2 + a_1^2 b_1^2 H_a R$ and $\rho \equiv R^6 + a_1^2 b_1^2 R^5$ are $a_0^6 a_1^6 b_1^4 + a_0^2 b_0^4 a_1^3$ and $a_0^6 b_1^5 + a_0^2 b_0^4 a_1^4 b_1$; subtracting $a_1^i b_1^j h$ and $b_1^{i+1} \rho$ $(i, j \leqq 6)$, we make the coefficient of a_0^6 in D_{60} a function of a_1 only; subtracting $a_1^i (b_1^7 - 1) h$, we make the coefficient of $a_0^2 b_0^4 b_1^7$ a constant.

The following invariants, free of a_2^6, have the indicated coefficients of b_2^6:

$$H_b R:\ a_0^2 b_0^4 b_1^6;\quad r \equiv H_b R^2 + a_1^2 b_1^2 H_b R:\ a_0^4 b_0^2 b_1^3 + b_0^6 a_1^4 b_1^6;$$
$$a_1^6 (R^5 + a_1^4 b_1^4 R^3) + b_1 H_a R:\ a_0^2 b_0^4 a_1^7;\quad a_1 \rho + a_1^2 b_1 h:\ b_0^6 a_1^6 + a_0^4 b_0^2 a_1^2 b_1^4.$$

Subtracting the product of the first by $a_1^r b_1^i$, the third by a_1^i $(i \leqq 6, r \leqq 7)$, we make the coefficient of $a_0^2 b_0^4$ in D_{06} a constant. Subtracting the product of the second by $a_1^i b_1^j$, the fourth by a_1^i $(i, j \leqq 6)$, we make the coefficient of b_0^6 in D_{06} a function of b_1. Subtracting $b_1^j (a_1^7 - 1) r$, we make the coefficient of $a_0^4 b_0^2 a_1^7$ a constant.

The following invariants are free of a_2^6 and b_2^6:

$$H_a H_b,\quad H_a R^4 + a_1^4 b_1^4 h,\quad H_b R^4 + a_1^4 b_1^4 r,$$

and have $a_0^4 b_0^4 a_1^6 b_1^6$, $a_0^4 b_0^4 a_1^7$, $a_0^4 b_0^4 b_1^7$ as coefficients of $a_2^4 b_2^4$. Subtracting their products by $a_1^i b_1^j$, a_1^i, b_1^i $(i, j \leqq 6)$, we make the coefficient of $a_0^4 b_0^4$ in D_{44} a constant, necessarily zero by (144_2).

By (144_8) and $(144_8')$, $b_0^4 D_{44}$ and $a_0^4 D_{44}$ involve only the $a_0^i b_0^j$ given by (143). Also $a_0^4 b_0^4$ has been deleted. Hence

$$D_{44} = h_2 a_0^4 + h_3 a_0^2 + h_4 a_0 + h_4 b_0^4 + h_6 b_0^2 + h_7 b_0' + h_8 .$$

Then, since D_{60} lacks b_0^6, (146_1) gives $b_1 h_8 = 0$ and

$$D_{60} = b_1^4 h_4 a_0^6 + d_1 a_0^4 b_0^2 + d_2 a_0^2 b_0^4 + d_3 a_0^2 b_0^2 + b_1^4 h_2 a_0^2 + d_4 a_0 b_0^2 + d_5 b_0^4 + d_6 b_0^3 + d_7 b_0^2 + b_1^4 h_3 ,$$
$$D_{42} = d_1 a_0^6 + d_2 a_0^4 b_0^2 + d_3 a_0^4 + d_4 a_0^3 + d_5 a_0^2 b_0^2 + d_6 a_0^2 b_0 + d_7 a_0^2 + b_1^4 h_5 b_0^2 + b_1^4 h_7 b_0^6 + b_1^4 h_6 .$$

Similarly, $(146_1')$ gives $a_1 h_8 = 0$ and

$$D_{24} = a_1^4 h_4 a_0^6 + e_1 a_0^4 b_0^2 + e_2 a_0^2 b_0^4 + e_3 a_0^2 b_0^2 + a_1^4 h_2 a_0^2 + e_4 a_0 b_0^3 + e_5 b_0^6 + e_6 b_0^4 + e_7 b_0^3 + e_8 b_0^2 + a_1^4 h_3 ,$$
$$D_{06} = e_1 a_0^6 + e_2 a_0^4 b_0^2 + e_3 a_0^4 + e_4 a_0^3 + e_5 a_0^2 b_0^4 + e_6 a_0^2 b_0^2 + e_7 a_0^2 b_0 + e_8 a_0^2 + a_1^4 h_5 b_0^2 + a_1^4 h_7 b_0^6 + a_1^4 h_6 .$$

In view of the above simplification of D_{60} and D_{06} by subtracting invariants, $b_1^4 h_4 = 0$, $a_1^4 h_7 = 0$, d_1 and e_5 are constants, the coefficient of b_1^7 in d_2 and that of a_1^7 in e_2 are constants. The second members of (144_3) and $(144_8')$ must involve only the $a_0^i b_0^j$ in (143). Hence

$$e_1 = e_3 = e_4 = e_6 = e_7 = 0, \quad d_3 = d_4 = d_5 = d_6 = 0, \quad e_2 = b_1^4 h_7 , \quad d_2 = a_1^4 h_4 .$$

Since $b_1^4 h_4 = 0$, $h_4 = A(b_1^7 - 1)$, A being a function of a_1 only. Then $d_2 = a_1^4 A(b_1^7 - 1)$. But the coefficient of b_1^7 in d_2 is a constant. Thus $a_1^4 A = 0$, so that A is a constant multiple of $a_1^7 - 1$. Hence $h_4 = c\pi$, π defined by (132). Similarly, $h_7 = k\pi$, where c and k are constants. Thus $e_2 = 0$, $d_2 = 0$. Also, $h_8 = l\pi$, l a constant. Since the terms of left member of (144_8) are multiples of a_1 or b_1, the constants d_1 and e_5 vanish; in fact, each enters a single term on the right. Hence

$$D_{60} = b_1^4 h_2 a_0^2 + d_7 b_0^2 + b_1^4 h_3 , \quad D_{42} = d_7 a_0^2 + b_1^4 h_5 b_0^2 + b_1^4 h_6 ,$$
$$D_{24} = a_1^4 h_2 a_0^2 + e_8 b_0^2 + a_1^4 h_3 , \quad D_{06} = e_8 a_0^2 + a_1^4 h_5 b_0^2 + a_1^4 h_6 .$$

Then (146_2) and $(146_2')$ give

$$b_0^2 D_{22} = b_0^2 \sigma, \quad a_0^2 D_{22} = a_0^2 \sigma, \quad \sigma = a_1^4 d_7 + b_1^4 e_8 .$$

Hence $D_{22} = \sigma + m(a_0^7 - 1)(b_0^7 - 1)$. But $a_0^7 b_0^7$ does not occur. Hence $D_{22} = \sigma$. The terms of the left members of (145_1), $(145_1')$ are multiples of a_0 or b_0. Hence $b_1 D_{22} = 0$, $a_1 D_{22} = 0$, $D_{22} = d\pi$, where d is a constant. Then $\sigma = d\pi$ gives $d = 0$.

In view of (143) and $D_{22} = 0$, (145_1) and $(145_1')$ show that D_{30} and D_{12} are linear homogeneous functions of $a_0^4 b_0$, $a_0^2 b_0$, $a_0 b_0^2$, $a_0 b_0$, b_0^5, b_0^3, b_0^2, b_0; and

D_{03}, D_{21} of $a_0 b_0^4$, $a_0 b_0^2$, $a_0^2 b_0$, $a_0 b_0$, a_0^5, a_0^3, a_0^2, a_0. But the right members of (146_3) and $(146_3')$ must involve only (143). Hence

$$D_{30} = \alpha a_0^2 b_0 + \beta b_0^3 + \gamma b_0, \quad D_{21} = \alpha a_0^3 + \beta a_0 b_0^2 + \gamma a_0,$$
$$D_{12} = \beta a_0^2 b_0 + \mu b_0^3 + \nu b_0, \quad D_{03} = \beta a_0^3 + \mu a_0 b_0^2 + \nu a_0.$$

The left members of (144_3) and $(144_3')$ are now of degree $\leqq 3$ in a_0, b_0; their right members of degree $\geqq 4$. Hence each member is zero. Thus

$$d_7 = e_8 = 0, \quad \alpha, \beta, \gamma, \mu, \nu, \quad h_2, h_3, h_5, h_6$$

are constant multiples of π.

In particular, D_{60}, D_{42}, D_{24}, D_{06} now vanish. By (144_2), every term of $b_0^4 D_{44}$ must be a multiple of a_1 or b_1; but each h_i is a constant multiple of π. Hence $D_{44} = 0$. Similarly, by (146_3), α, γ, μ, ν vanish. After subtracting βR^5, we have $\beta = 0$, since

$$\pi R^5 = \pi (a_2^2 b_0^2 + b_2^2 a_0^2)(a_2 b_0 + b_2 a_0).$$

Thus D_{30}, D_{21}, D_{12}, D_{03} now vanish. Then $D_{11} = 0$ by (145_2), $(145_2')$. In view of the eleven D's just proved zero and (143), the possible factors $a_0^i b_0^j$ are now

$$i, j = 50,\ 41,\ 40,\ 20,\ 14,\ 10,\ 05,\ 04,\ 02,\ 01,\ 00. \tag{147}$$

Hence by (144_1) and $(144_1')$, D_{50} and D_{41} involve only 41, 14, 05, 04; D_{14} and D_{05} only 50, 41, 40, 14. Applying also (145_3) and $(145_3')$, we get

$$D_{50} = s b_0^5, \quad D_{41} = s a_0 b_0^4, \quad D_{14} = s a_0^4 b_0, \quad D_{05} = s a_0^5.$$

By (144_2), s is a multiple of π. But

$$\pi R^6 = \pi (a_2 b_0 + b_2 a_0)(a_2^4 b_0^4 + b_2^4 a_0^4).$$

Hence by subtracting sR^6, we make $s = 0$. Thus by (147),

$$i, j = 40,\ 20,\ 10,\ 04,\ 02,\ 01,\ 00 \tag{148}$$

give the only non-vanishing D_{ij} and the possible factors $a_0^i b_0^j$. Then T_{k00}, $k = 1, 2, 4$, give

$$\begin{aligned} a_1 D_{10} + b_1 D_{01} = a_0^4 D_{40} + b_0^4 D_{04}, \quad a_1^2 D_{20} + b_1^2 D_{02} = a_0 D_{10} + b_0 D_{01}, \\ a_1^4 D_{40} + b_1^4 D_{04} = a_0^2 D_{20} + b_0^2 D_{02}. \end{aligned} \tag{149}$$

The second members must involve only the $a_0^i b_0^j$ given by (148). Hence

$$D_{40} = p a_0^4 + q b_0^4 + r, \quad D_{10} = P a_0 + Q b_0 + U, \quad D_{20} = \rho a_0^2 + \sigma b_0^2 + \lambda,$$
$$D_{04} = q a_0^4 + s b_0^4 + t, \quad D_{01} = Q a_0 + S b_0 + T, \quad D_{02} = \sigma a_0^2 + \mu b_0^2 + \nu.$$

Then relations (149) are satisfied if, and only if,

$$r=t=U=T=\lambda=\nu=0,\quad p=a_1P+b_1Q,\quad s=a_1Q+b_1S, \tag{150}$$
$$P=a_1^2\rho+b_1^2\sigma,\quad S=a_1^2\sigma+b_1^2\mu,\quad \rho=a_1^4p+b_1^4q,\quad \mu=a_1^4q+b_1^4s. \tag{151}$$

Subtracting $a_1^i b_1^r H_a$, $b_1^{i+1}R^4$ ($i\leqq 6$, $r\leqq 7$, we have $p=$ constant. Subtracting $a_1^r b_1^s R^2$, we have $q=0$. In $a_1(R^4+a_1^4b_1^4R^2)+a_1^3b_1H_a$ and H_b, the coefficients of a_2^4 are zero; those of b_2^4 are $b_0^4a_1^3$ and $b_0^4b_1^6$. Subtracting the product of the first by a_1^4, the second by $a_1^r b_1^t$, we make $s=$ constant. Subtracting $a_1^r b_1^s R$, which is free of a_2^4, b_2^4, we make $\sigma=0$. Then (151) becomes

$$\rho=pa_1^4,\quad \mu=sb_1^4,\quad P=pa_1^6,\quad S=sb_1^6.$$

The final conditions (150) give

$$b_1Q=p(a_1^7+1),\quad a_1Q=s(b_1^7+1).$$

Since p and s are constants, we have $p=s=0$, $Q=c\pi$. Hence

$$\phi=c\pi(a_2b_0+b_2a_0)+D_{00}.$$

We make $c=0$ by subtracting $c\pi R^4$, since

$$\pi R^4=\pi(a_2b_0+b_2a_0).$$

Hence ϕ is now D_{00} and therefore by (56), a function of a_1, b_1 only.

In view of the last two foot-notes, it is readily seen that the invariants, which have been subtracted from ϕ to reduce it to $\Sigma a_1^r b_1^s$, together with the latter, are linearly equivalent to the set (82).

THEOREM. *Every invariant of a pair of quadratic forms in the* $GF[2^3]$ *is an integral function of the eight independent invariants* (81); *indeed a linear combination of the linearly independent invariants* (82), *for* $n=3$.

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Surfaces and Congruences Derived from the Cubic Variety Having a Double Line in Four-Dimensional Space.

BY VIRGIL SNYDER.

Twenty years ago Professors Castelnuovo* and Segre† published a series of memoirs on the cubic varieties of ∞^3 points in space of four dimensions S_4 which have been the foundation of many important researches since.‡

The most important ideas are those of the apparent contour (section of the enveloping cone from a given point by ordinary space) and the systems of lines contained upon it which project into bitangents of the apparent contour. The treatment in Segre's longer paper is entirely synthetic; many particular cases are mentioned, but they are not considered in detail. The purpose of the present paper is to extend the results given in the second part of Segre's memoir and to show the connection with a number of known configurations. Part of the results of every article of the present paper except the last are given by Segre, but the method is analytic and no knowledge of the previous papers will be assumed.

§ 1. *One Double Line.*

1. The most general cubic variety Γ in S_4 having a double line d defined by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ may be expressed in the form

$$\Gamma \equiv ax_1^2 + hx_1x_2 + bx_2^2 + gx_1x_3 + fx_2x_3 + cx_3^2 = 0, \tag{1}$$

* "Sopra una congruenza del 3° ordine e 6ª classe dello spazio a quattro dimensioni e sulle sue proiezioni nello spazio ordinario," *Atti del Istituto Veneto*, Ser. 6, Vol. V (1887), pp. 1249–1281, and Vol. VI (1888), pp. 525–579.

† "Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario," *Memorie di Torino*, Ser. 2, Vol. XXXIX (1889), pp. 1–48, and "Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni," *Atti di Torino*, Vol. XXII (1887), pp. 791–801.

‡ Particularly by G. Fano, in the *Atti Ist. Veneto* (1896), and *Atti di Torino* (1904); A. Dragoni, in *Batt. Giornale* (1902); and H. W. Richmond, in *Quar. Jour.* (1903).

in which $a \equiv \Sigma a_i x_i$, and similarly for the other coefficients. The tangent S_3 at any point as $(0, 0, 0, t_4, t_5)$ on d is indeterminant. The second polar M_2 has the equation

$$t_4(a_4 x_1^2 + \ldots) + t_5(a_5 x_1^2 + \ldots) = 0, \qquad (2)$$

which always contains d. When the point describes d, M_2 forms a pencil, the four basis planes passing through d. Three polars of the pencil break up into pairs of spaces, corresponding to those values of $t_4 : t_5$ for which the discriminant of (2) vanishes. These points are bispatial on d. The six associated tangent spaces define by proper pairs the basis planes of the pencil.

A general S_3 will cut Γ in a cubic surface having a double point where S_3 cuts d. Through this point pass six lines lying on the surface, which contains 15 other lines. The plane formed by joining any point P on Γ to d will cut Γ in d counted twice, and in a line through P. The lines of Γ cutting d define a (1, 6) system. The S_3 tangent to Γ at P will have a double point at P and another at the intersection of S_3 with d. The line joining them is counted twice, hence through P pass 4 lines not cutting d. The residual system of lines on Γ not meeting d is (4, 15).

The apparent contour from a point on Γ will have d for a double line; it will also have a double plane containing four other double points on its conic of contact. The surface F_4 is focal for a (2, 6) congruence having a focal line, and a congruence (8, 15). Any plane through the projection of d will cut F_4 in a conic. The two tangents to this conic from any point of its plane are the lines of the (2, 6) congruence. Any plane π not containing d will cut d in a point Π. Through Π in π can be drawn six tangents to six different conics of the system. On d are three cusps, images of the three bispatial points in S_4, and a fourth, image of the point of intersection of d with the polar S_3 of the center of projection.

When the center of projection is not on Γ, the apparent contour will be a surface F_6 of order 6, class 12, with a cuspidal sextic c_6 and a double line. The congruence of lines tangent to the surface and meeting d is (3, 6). Any plane through d cuts F_6 in a quartic curve having three cusps. The point Π is now a uniplanar triple point on F_6. It is a double point on c_6, with d for a nodal tangent. If the center of projection is in one of the tangent planes touching Γ along d, the cubic covariant G is a ruled surface having d for double directrix. The Hessian will also contain d; hence, from the equation

$$F_6 \equiv 4H^3 + G^2 = 0,$$

d is a triple line on the surface. The residual cuspidal curve is a rational c_4 lying on H, having d for trisecant. Any plane through d cuts F_6 in a cuspidal cubic. When the simple directrix of G lies on H, c_4 breaks up into this directrix and three generators. In this case

$$\Gamma \equiv x_3^3 + x_1 x_2 x_4 + x_5 (x_1^2 + x_2^2 - x_3^2) = 0.$$

The four singular tangent planes are $x_1 = 0,\ x_2 \pm x_3 = 0$; $x_2 = 0,\ x_1 \pm x_3 = 0$; and the three bispatial points on d are $(0, 0, 0, 1, 0)$, $(0, 0, 0, 2, 1)$, $(0, 0, 0, 2, -1)$. The apparent contour from $(0, 1, 1, 0, 0)$ on its polar S_3 or $x_3 = 0$ is

$$27\,[x_1 x_2 x_4 + x_5 (x_1^2 + x_2^2)]^2 + 4\,[x_1 x_4 + 2x_2 x_5]^3 = 0.$$

The triple line is $x_1 = 0,\ x_2 = 0$; and the four cuspidal lines are $x_4 = 0,\ x_5 = 0$; $x_1 = 0,\ x_5 = 0$; $x_1 + x_2 = 0,\ x_4 = 2x_5$; $x_1 - x_2 = 0,\ x_4 + 2x_5 = 0$. Of these the first is the simple directrix and the last two are torsal generators of G. The triple line is inflexional tangent for the cubic curve in every plane through it.

2. If the entire plane $x_1 = 0,\ x_2 = 0$ lies on Γ, then $c \equiv 0$. The point $x_1 = 0,\ x_2 = 0,\ f = 0,\ g = 0$ is now a double point on Γ. Let the plane be π_1 and the double point D_1. The pencil $x_2 = kx_1$ through π_1 will cut Γ in π_1 and a pencil of quadrics

$$(a + 2hk + bk^2)\, x_1 + 2x_3 (kf + g) = 0$$

which contains d and a variable line passing through D_1. Conversely, if Γ contains a double point not lying on d, the plane containing it and d will lie entirely on Γ.

The plane $x_1 = 0,\ x_2 = 0$ is now one of the basis planes touched by all the tangent hypercones having the entire line d as vertex and belonging to the points of d. Such hypercones are called hypercones of the second species.

The lines of Γ form three systems: the first $(1, 5)$ composed of the lines cutting d; the second $(1, 5)$ composed of the lines cutting π_1, but not cutting d; and a $(3, 10)$ cutting neither. The second consists of the generators of the quadrics, of the same system as d, while the first is composed of the generators of the other system.

The contours F_4, F_6 are now of class 10, have a double point, and are focal surfaces of congruences $(2, 5)$, $(3, 5)$.

3. If Γ also contains the plane $x_1 = 0,\ x_3 = 0$, or π_2, then $b \equiv 0$. We may write $f \equiv x_4,\ g \equiv x_5$ without loss of generality. The point $h = 0,\ x_4 = 0$ or D_2 is also a double point. The space $x_1 = 0$ cuts Γ in the planes π_1, π_2 and

the plane $x_4 = 0$ or σ_3, passing through both double points and meeting d in $(0, 0, 0, 0, 1)$. The pencil $kx_1 = x_4$ through π_2 cuts Γ in the system of quadrics, whose section by $x_1 = 0$ is the pencil of conics

$$x_3 x_5 + (h_2 x_2 + h_3 x_3 + h_5 x_5) x_2 + kx_2 x_3 = 0.$$

These conics all pass through D_1, D_2 and touch each other on d, the common tangent being $h_5 x_2 + x_3 = 0$. It is a bispatial point on d.

The lines of Γ are now the $(1, 4)$ meeting d, the $(1, 4)$ meeting π_1 but not d, the $(1, 4)$ meeting π_2, the $(2, 6)$ meeting σ_3. Γ can be generated by the trilinear system

$$\begin{aligned} &\alpha x_1 - \beta x_3 = 0, \\ &\alpha x_5 + \beta \bar{a} + \gamma x_2 = 0, \\ &\alpha x_4 + \beta(a_2 x_1 + h) - \gamma_1 = 0, \end{aligned}$$

and its conjugate, where $\bar{a} = a - a_2 x_2$.

4. If Γ also contains the plane $x_2 = 0$, $x_3 = 0$ or π_3, then $a \equiv 0$. There is also a double point D_3 at $h = 0$, $x_5 = 0$ in π_3. The variety now contains six planes $\pi_1, \pi_2, \pi_3, \sigma_1, \sigma_2, \sigma_3$ and three double points D_1, D_2, D_3. Each plane σ_i cuts d in a bispatial point and contains two double points. Γ has just one proper tangent plane touching it throughout d.

The lines of Γ are arranged in five systems, all of form $(1, 3)$. The first is composed of the lines cutting d, the next three cut $\pi_i \sigma_i$, and the last cuts $\sigma_1 \sigma_2 \sigma_3$.

From a point P on Γ the apparent contour is a surface of order 4, class 6, having a double line with four cuspidal points, four double planes with four double points besides the intersection of each with d, and three other double points. It is complete focal surface for four $(2, 3)$ congruences, and the double line is focal line for another, the surface being the other sheet of the focal surface.

From a point not on Γ the apparent contour will be of order and class 6, having a cuspidal sextic, a double line and three other double points. If the point be taken in the plane $D_1 D_2 D_3 = \tau$, the double points will be collinear. The only proper tangent plane containing d is

$$h_4 x_1 + x_3 = 0, \quad h_5 x_2 + x_3 = 0.$$

The equations of τ are

$$h_1 x_1 + h_4 x_4 = 0, \quad h_2 x_2 + h_5 x_5 = 0.$$

From the point of intersection of these two planes $(-h_5^2 h_4, -h_4^2 h_5, h_4^2 h_5^2, h_4^2 h_2, h_5^2 h_1)$ F_6 will have a triple line and three collinear double points. If the center of projection be taken in τ on Γ, a double point becomes uniplanar, and one conic of contact in a double plane is two coincident lines.

5. Finally, it is possible that

$$\Gamma \equiv h x_1 x_2 + x_2 x_3 x_4 + x_1 x_3 x_5 = 0$$

contain another plane passing through d. Its equations may be taken

$$x_1 = x_2 = x_3.$$

It will lie on Γ if $h_4 = -1$, $h_5 = -1$, $h_1 + h_2 + h_3 = 0$, and there will be a double point $(1, 1, 1, h_1, h_2)$ or D_4 lying in π_4.

$$\Gamma \equiv x_2 (x_1 - x_3)(h_1 x_1 - x_4) + x_1 (x_2 - x_3)(h_2 x_2 - x_5) = 0. \tag{3}$$

The four double points are:

$$D_1 \equiv (0, 0, 1, 0, 0), \qquad D_3 \equiv (1, 0, 0, h_1, 0),$$
$$D_2 \equiv (0, 1, 0, 0, h_2), \qquad D_4 \equiv (1, 1, 1, h_1, h_2);$$

they all lie in the plane τ

$$x_5 - h_2 x_2 = 0, \quad x_4 - h_1 x_1 = 0,$$

which from (3) lies on Γ.

Besides the planes τ, π_i, Γ contains 6 planes σ_{ik} defined as the residual intersection with Γ of the space containing the planes π_i, π_k. The equations of the complete set of eleven planes are:

$$\begin{array}{lll}
\pi_1: & x_1 = 0, & x_2 = 0. \\
\pi_2: & x_1 = 0, & x_3 = 0. \\
\pi_3: & x_2 = 0, & x_3 = 0. \\
\pi_4: & x_1 - x_3 = 0, & x_2 - x_3 = 0. \\
\sigma_{12}: & x_1 = 0, & x_4 = 0. \\
\sigma_{13}: & x_2 = 0, & x_5 = 0. \\
\sigma_{14}: & x_1 - x_2 = 0, & h_1 x_1 + h_2 x_2 - x_4 - x_5 = 0. \\
\sigma_{23}: & x_3 = 0, & h_1 x_1 + h_2 x_2 - x_4 - x_5 = 0. \\
\sigma_{24}: & x_1 - x_3 = 0, & h_2 x_2 - x_5 = 0. \\
\sigma_{34}: & x_2 - x_3 = 0, & h_1 x_1 - x_4 = 0. \\
\tau: & h_1 x_1 - x_4 = 0, & h_2 x_2 - x_5 = 0.
\end{array}$$

The planes π_i contain each one double point D_i and d; the planes σ_{ik} contain two double points D_i, D_k and cut d in bispatial points; the plane τ

passes through all four double points and does not meet d. Thus, through each point D_i pass five planes τ, π_i, σ_{ih}. ($h \neq i$.)

6. The variety Γ may be generated by the following trilinear systems:

$$\text{I}\begin{cases} \alpha x_2 + \beta(x_2 - x_3) = 0, \\ \beta(x_1 - x_3) + \gamma(h_2 x_2 - x_5) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha x_1 = 0. \end{cases} \qquad \text{II}\begin{cases} \alpha x_2 + \beta(h_2 x_2 - x_5) = 0, \\ \beta(x_1 - x_3) + \gamma x_1 = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(x_2 - x_3) = 0. \end{cases}$$

$$\text{III}\begin{cases} \alpha x_2 + \beta x_1 = 0, \\ \beta(x_1 - x_3) + \gamma(h_2 x_2 - x_5) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(x_2 - x_3) = 0. \end{cases} \qquad \text{IV}\begin{cases} \alpha x_2 + \beta(h_2 x_2 - x_5) = 0, \\ \beta(x_1 - x_3) + \gamma(x_2 - x_3) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha x_1 = 0. \end{cases}$$

$$\text{V}\begin{cases} \alpha x_2 + \beta x_1 = 0, \\ \beta(x_1 - x_3) + \gamma(x_2 - x_3) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(h_2 x_2 - x_5) = 0. \end{cases}$$

The system V consists of lines cutting d and τ. The pencil of spaces through τ will cut d in a projective range. The quadric surface in the residual section with Γ will have a double point on d; hence every surface of the system will be a cone. In case of the three spaces through τ:

$$\begin{aligned} h_1 x_1 - x_4 &= 0, \\ h_2 x_2 - x_5 &= 0, \\ h_1 x_1 - x_4 - (h_2 x_2 - x_5) &= 0, \end{aligned}$$

the quadric cone breaks up into a pair of planes. These are

$$\begin{matrix} \sigma_{23}, & \sigma_{14}, \\ \sigma_{13}, & \sigma_{24}, \\ \sigma_{12}, & \sigma_{34}, \end{matrix}$$

respectively. The points on d on the lines of intersection of these planes are bispatial points:

$(0, 0, 0, 0, \ \ 1)$, with the tangent spaces $x_1 = 0$, $x_2 - x_3 = 0$,
$(0, 0, 0, 1, \ \ 0)$, " " " " $x_2 = 0$, $x_1 - x_3 = 0$,
$(0, 0, 0, 1, -1)$, " " " " $x_3 = 0$, $x_1 - x_2 = 0$,

respectively. The four pencils $\pi_i D_i$ do not belong to V.

The lines of I cut π_1, σ_{24}, σ_{23}, σ_{34}. A pencil of spaces through π_1 will cut Γ in a series of quadrics, whose section by π_1 is d and a line through D_1. The lines of I do not cut d, hence belong to the same system of generators of these

quadrics as d. In general the pencils σ_{ik}, D_k belong to (i). There are no proper quadric cones on Γ with vertices at D_4. The sextic cones at the double points break up into plane pencils. All of these systems are (1, 2).

7. The apparent contour from a point on Γ will be a surface of order and class 4, having a double line, two double planes and four double points. It is the general complex surface of Plücker, formed by the lines of a quadratic complex which cut a given line not belonging to the complex. Its symbol is [21111].*

If, however, the center of projection be taken in τ, all the four double points will project into collinear points, and the line joining them will be a double line. The lines of V now project into lines cutting two skew double lines, hence F_4 is a ruled surface contained in a linear congruence. From $(1, 2, 3, h_1, 2h_2)$ on $x_5 = 0$ the contour becomes

$$[2(x_1 - x_2 - x_3)(h_1 x_1 - x_4) + h_2 x_2 (x_2 - x_3 - x_1)]^2$$
$$+ 4[x_2(x_1 - x_3)(h_1 x_1 - x_4) + h_2 x_1 x_2 (x_2 - x_3)]\,[4(h_1 x_1 - x_4) + h_2 x_2] = 0.$$

The double directrices are $x_3 - 3x_1 = 0$, $x_2 - 2x_1 = 0$; $x_2 = 0$, $h_1 x_1 - x_4 = 0$. The symbol is [(11)1111].

8. The apparent contour from a point not on Γ is a surface of order 6, class 4, having a double line, a cuspidal sextic curve and four double points. An interesting case arises when $h_1 = h_2 = 0$. If we project from (0, 1, 1, 1, 0) on the polar space $x_1 = x_2 + x_3 + x_4$ the equation of the focal surface becomes

$$4[(x_2 + x_4)^2 - x_2 x_4]^3 = 27[x_2 x_4 (x_2 + x_4) + x_1 x_5 (2x_2 - x_1 + x_4)]^2.$$

The double line becomes $x_1 = 0$, $2x_2 + x_4 = 0$, and the cuspidal curve is composed of the two plane cubics

$$x_2 = \omega x_4, \qquad x_2^3 = x_1 x_5 ((2 + \omega) x_2 - x_1), \qquad [\omega^3 = 1$$

each of which has a node at $(x_1, x_2, x_4, x_5) \equiv (0, 0, 0, 1)$ on the double line. Any plane section through the double line will consist of a quartic having three cusps, one of which is at (0, 0, 0, 1). There are but five distinct double planes:

$$\begin{aligned}
\pi_1 = \pi_2 = {}& \sigma_{12} \equiv x_1 = 0,\\
\pi_3 = \pi_4 = {}& \sigma_{34} \equiv 2x_2 - x_1 + x_4 = 0,\\
\sigma_{13} = \sigma_{24} = {}& \tau \equiv x_5 = 0,\\
\sigma_{14} \equiv {}& x_1 - x_2 + x_4 + x_5 = 0,\\
\sigma_{23} \equiv {}& x_2 + 2x_4 + x_5 - x_1 = 0.
\end{aligned}$$

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* A. Weiler: "Ueber die verschiedenen Gattungen der Complexe zweiten Grades," *Math. Ann.*, Vol. VII (1874), or the corrected list given by Jessop, *Treatise on the Line Complex*, p. 230.

Three double planes pass through each double point. The sections made by the double planes σ_{12}, σ_{34}, τ consist of three concurrent lines, each counted twice. Thus, through $(0, 0, 0, 1)$ pass the double line and four simple lines of the surface, the latter having fixed tangent planes throughout.

The bitangents of the surface can be readily arranged in four systems of quadrics, each of index 3. Thus III becomes

$$\frac{\alpha\beta(2x_3 + x_4) + \beta^2 x_1}{\alpha x_5} = \frac{\alpha^2(x_1 - 2x_3 - x_4)}{\alpha(x_3 - x_4) + \beta x_1}.\,*$$

§ 2. *Two Intersecting Double Lines.*

9. The variety Γ may have the line d' defined by $x_1 = 0$, $x_2 = 0$, $x_4 = 0$ for double line, as well as d. The general equation is

$$\Gamma \equiv a x_1^2 + h x_1 x_2 + b x_2^2 + x_1 x_3 x_4 = 0.$$

The tangent S_3 to Γ at any point $(0, 0, \xi_3, \xi_4, \xi_5)$ lying in the plane $x_1 = 0$, $x_2 = 0$ or π has the equation $x_1 = 0$. The residual intersection of $x_1 = 0$ with Γ is the plane σ defined by $b = 0$, cutting d in the point $(0, 0, 0, -b_5, b_4)$ and d' in the point $(0, 0, b_5, 0, -b_3)$. The polar M_2 of the first point is

$$b_5(a_4 x_1^2 + h_4 x_1 x_2 + b_4 x_2^2 + x_1 x_3) - b_4(a_5 x_1^2 + h_5 x_1 x_2 + b_5 x_2^2) = 0,$$

which factors into x_1 and another linear factor; hence the point is bispatial. The point on d' is also bispatial, $x_1 = 0$ being common to both. The $M_2^{(3)}$ of $(0, 0, 0, 0, 1)$, the point of intersection of d, d', is $a_5 x_1^2 + h_5 x_1 x_2 + b_5 x_2^2 = 0$. It consists of two spaces, both passing through π. The spaces of the pencil $x_2 = kx_1$ passing through π cut Γ in a series of quadrics

$$x_1(a + kh + k^2 h) + x_3 x_4 = 0$$

cutting π in d, d'. The systems of lines cutting d, d' are each $(1, 4)$.

A pencil $x_1 = \lambda b$ through σ cuts Γ in a series of quadrics whose section with σ is a pencil of conics touching each other at the bispatial points on d, d'. This system is $(2, 6)$, the class being obtained as follows. Any S_3 cuts Γ in a cubic surface having two double points, containing 16 lines made up of the line joining the two double points, 4 others through each, and 7 not passing through either. Of the latter, one is the intersection with σ.

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* For a similar particular case of a variety having nine distinct double points see my paper in the *Trans. Amer. Math. Soc.*, Vol. X (1909). The necessary and sufficient condition that the cuspidal sextic breaks up into two plane cubics is that the Hessian is the product of two spaces. This will happen when the center of projection lies on a definite curve of S_4.

10. Now suppose Γ has a double point not lying in π nor in σ. $D_1 \equiv (1, 0, 0, 0, 0)$. The plane determined by D_1 and d must lie entirely on Γ. This is $x_2 = 0$, $x_3 = 0$; hence $a = 0$. Similarly, the plane of D_1 and d' lies on Γ, but $x_2 = 0$, $x_4 = 0$ already satisfies the equation.

$$\Gamma \equiv hx_1 x_2 + bx_2^2 + x_1 x_3 x_4 = 0.$$

The system of lines cutting d is now $(1, 3)$. Its equations are

$$\begin{aligned} \alpha x_2 - \beta x_3 &= 0, \\ \alpha x_4 + h\beta + \gamma b &= 0, \\ \beta x_2 - \gamma x_1 &= 0. \end{aligned}$$

The $(1, 3)$ system cutting d' has the equations

$$\begin{aligned} \alpha x_2 - \beta x_4 &= 0, \\ \alpha x_3 + \beta h + \gamma b &= 0, \\ \beta x_2 - \gamma x_1 &= 0. \end{aligned}$$

The conjugate of the first system cuts σ and the plane $D_1 d'$. The conjugate of the second cuts σ and the plane $D_1 d$.

11. If Γ has another double point D_2 at $(1, 1, 0, 0, 0)$, and $b \equiv x_5$,

$$\Gamma \equiv (x_1 - x_2)\, x_2 x_5 + x_1 x_3 x_4 = 0.$$

The plane σ and the line $D_1 D_2$ lie in the space $x_5 = 0$ which cuts Γ in σ and two new planes

$$\tau : x_3 = 0,\ x_5 = 0\,; \qquad \tau_2 : x_4 = 0,\ x_5 = 0,$$

which intersect in the line $D_1 D_2$. The first passes through the bispatial point on d, the second through the bispatial point on d'. The eight planes on Γ are now the following:

$$\begin{array}{ll} d,\ d' : & x_1 = 0,\ x_2 = 0, \\ D_1 d : & x_2 = 0,\ x_3 = 0, \\ D_1 d' : & x_2 = 0,\ x_4 = 0, \\ D_2 d : & x_1 - x_2 = 0,\ x_3 = 0, \\ D_2 d' : & x_1 - x_2 = 0,\ x_4 = 0, \\ \sigma : & x_1 = 0,\ x_5 = 0, \\ \tau_1 : & x_3 = 0,\ x_5 = 0, \\ \tau_2 : & x_4 = 0,\ x_5 = 0. \end{array}$$

There are now four systems $(1, 2)$:

I	cuts	$D_2 d$, $D_1 d'$, σ.
II	"	τ_1, d'.
III	"	τ_2, d.
IV	"	$D_1 d$, $D_2 d'$, σ.

The first and third are conjugate, as are the second and fourth.

12. The apparent contour from $(0, 1, 0, 0, 1)$ on its polar S_3 is

$$4[3x_2^2 - 3x_1 x_2 + x_1^2]^3 - 27[(x_1 - x_2)(x_1 - 2x_2) x_2 + x_1 x_3 x_4]^2 = 0.$$

It is of order 5, as x_1 appears as a factor. Any plane $x_2 - k\, x_1 = 0$ will cut the surface in the line $x_1 = 0$, $x_2 = 0$ and two conics which touch each other at $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, the common tangents being $x_3 = 0$, $x_4 = 0$. The planes $x_3 = 0$, $x_4 = 0$ are double planes, the conics of contact breaking up into two lines intersecting at the point of contact of the conics, which are both triple points on the surface. The plane $x_1 = 0$ contains the line joining the triple points as a threefold line. The points $(3, 2, 0, 0)$, $(3, 1, 0, 0)$ are both double. The planes $x_1 = k_1 x_2$, $x_1 = k_2 x_2$ cut the surface in conics which are cuspidal on the surface, k_1, k_2 being roots of the quadratic equation $3k^2 - 3k + 1 = 0$.

The planes (dd') and σ project into $x_1 = 0$.
" " $D_1 d$, $D_2 d$, τ_1 " " $x_3 = 0$.
" " $D_1 d'$, $D_2 d'$, τ_2 " " $x_4 = 0$.

The congruences II, III have focal lines; the other two interchange by the transposition $(x_3 x_4)$, an involution which leaves the surface invariant.

13. If the center of projection be taken upon Γ, the apparent contour is the Plücker complex surface with one more double line, having the symbol [1122]. The usual form of equation results when the center is at $(2, 4, 2, 2, 1)$, and an interesting particular case from $(1, 0, 0, 0, 1)$.

§ 3. *Three Concurrent Double Lines.*

14. If Γ has three double lines d, d', and d'', the last defined by $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, its equation becomes

$$\Gamma \equiv (ax_3 + bx_4)\, x_1 x_2 + x_2^2 x_5 + (cx_1 + dx_2)\, x_3 x_4 = 0.$$

The tangent S_3 at every point of the plane π_{12} containing the first two lines is $cx_1 + dx_2 = 0$. It cuts Γ in the residual plane $\sigma_3 : d(ax_3 + bx_4) - cx_5 = 0$. The tangent S_3 at points of $\pi_{13} : x_2 = 0$, $x_3 = 0$ is $bx_2 + cx_3 = 0$, and the residual plane is $\sigma_2 : b(ax_1 + dx_4) - cx_5 = 0$. For $\pi_{23} : x_2 = 0$, $x_4 = 0$ the tangent is $ax_2 + cx_4 = 0$, and the residual plane is $\sigma_1 : a(bx_1 + dx_3) - cx_5 = 0$. The point $(0, 0, 0, 0, 1)$ is unispatial, $x_2 = 0$ being the tangent S_3. On d', the planes σ_2 and σ_3 meet in the bispatial point $(0, 0, 0, c, bd)$, the tangent spaces being

$$c[bx_1 x_2 + x_3(cx_1 + dx_2)] + dbx_2^2 = 0.$$

The lines of Γ form three $(1, 2)$ systems, cutting d, σ_1; d', σ_2; d'', σ_3.

15. From a point not on Γ the apparent contour becomes an F_6 with a cuspidal c_8, three concurrent double lines and seven triple points. It is the dual of the cubic surface cut from Γ by a tangent S_3. Each of the three (3, 2) congruences has a focal line.

From a point on Γ, the contour becomes the Steiner surface of order 4, class 3, and symbol [222]. By taking the center at (0, 1, 1, 0, 0), the section by $x_2 = 0$ is the canonical form

$$x_1^2 x_3^2 + x_3^2 x_4^2 + x_4^2 x_1^2 - 2x_1 x_3 x_4 (2x_5 - x_3) = 0.$$

If the center of projection be taken in π_{ik}, [42] results.

§ 4. *One Double Line of the Second Species.*

16. A double line on Γ is said to be of the second species when the tangent $M_2^{(3)}$ at every point upon it breaks up into two spaces. If $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ be the double line, then

$$\Gamma \equiv ax_1^2 + hx_1 x_2 + bx_2^2 + Ax_3^2 = 0,$$

$A = \sum_{i=1}^{3} A_i x_i$. The tangent $M_2^{(3)}$ of $(0, 0, 0, \xi_4, \xi_5)$ is

$$\xi_4(a_4 x_1^2 + h_4 x_1 x_2 + b_4 x_2^2) + \xi_5(a_5 x_1^2 + h_5 x_1 x_2 + b_5 x_2^2) = 0.$$

Thus, the pairs of spaces have the plane $x_1 = 0$, $x_2 = 0$ in common, and form an involution projective with the point of tangency. The points on d belonging to the double elements of the involution will be unispatial. The section of Γ made by the space $x_2 = kx_1$ through the basis plane π will be

$$\phi_3(x_1, x_3) + [x_4 (a_4 + h_4 k + b_4 k^2) + x_5 (a_5 + h_5 k + b_5 k^2] x_1^2 = 0,$$

hence a cubic cone with vertex at $(0, 0, 0, a_5 + h_5 k + b_5 k^2, -(a_4 + h_4 k + b_4 k^2))$ having d for cuspidal edge, and π for tangent plane. The lines cutting d form a (1, 6) system; the residual system on Γ is (3, 9).

17. The apparent contour of Γ from a point upon it is a surface of order 4, class 6, with a cuspidal line having a fixed tangent plane and two triple points. The surface has a double plane, the conic of contact passing through two double points besides the cusp on d. It is focal surface of a (2, 6) and a (6, 9) congruence, the former having a focal line.

From a point not on Γ the contour is of order and class 6, having a cuspidal line and a cuspidal sextic with a point in common. This and the projection of the two unispatial points are triple points on the surface. The cuspidal line has a fixed tangent plane which cuts F_6 in three lines meeting at the uniplanar triple

point. In particular, if P be taken on π, its polar $M_2^{(3)}$ will be a hypercone having d for vertex, and its polar S_3 will always pass through d. Its equation is $A + 2A_3x_3 = 0$, independent of P. Hence every line of the (1, 6) system cutting d will go into a line lying entirely on the focal surface, a ruled surface of order 6, having d for multiple directrix. Its equation is of the form

$$[\phi_2(x_1, x_2)]^3 + K[x_5x_1^2 + x_4x_2^2 + hx_1x_2 + \phi_3(x_1, x_2)]^2 = 0.$$

Any plane $x_2 = kx_1$ through the fourfold line contains two generators which intersect upon it, the point being the same for two values of k. The generators passing through the images of the unispatial points are cuspidal. The surface is rational and has no other double line. It is form 101 in my enumeration.* An interesting particular case is that in which the double elements of the involution approach coincidence. The ruled sextic now has a tacnodal generator. Its equation is

$$(x_4x_2^2 - x_1^2x_3)^2 = x_1x_2^5.$$

18. Suppose Γ contains a double point not lying on d. This requires that $A \equiv 0$. Now any space $x_2 = kx_1$ passing through π will contain π as a squared factor, the other being

$$a + hk + bk^2 = 0;$$

hence Γ contains an infinite number of planes, each lying in an S_3 with π. Every point $(0, 0, \xi_3, \xi_4, \xi_5)$ of π is bispatial, the tangent M_2 being of the form

$$\xi_3(a_3x_1^2 + h_3x_1x_2 + b_3x_2^2) + \xi_4(a_4x_1^2 + \dots) + \xi_5(a_5x_1^2 + \dots) = 0.$$

The two spaces become coincident for points satisfying the equation

$$4(a_3\xi_3 + a_4\xi_4 + a_5\xi_5)(b_3\xi_3 + b_4\xi_4 + b_5\xi_5) = (h_3\xi_3 + h_4\xi_4 + h_5\xi_5)^2.$$

Thus, *if Γ has a double plane, there is a conic lying in the plane, locus of unispatial points.*

Since a space $x_2 = kx_1$ is uniquely determined by a point not in π, it follows that no two planes of Γ

$$x_2 = kx_1, \qquad a + hk + bk^2 = 0;$$
$$x_2 = lx_1, \qquad a + hl + bl^2 = 0$$

can lie in the same S_3, except when $h = 0$; but in this case Γ is a hypercone, projection of a ruled cubic surface of S_3 from a point outside. Excluding this case, it is seen that through any point $(0, 0, \xi_3, \xi_4, \xi_5)$ of π pass two lines, $a + hk + bk^2 = 0$, intersection of the generating planes with π. The envelope of these lines is the conic of unispatial points c_2.

* JOURNAL, Vol. XXVII (1905), pp. 76–102. See p. 95.

The tangent space T at a point P of c_2 will cut π in the tangent t to c_2 at P. It will cut Γ in a generator plane α and in a quadric surface. The quadric passes through t and another line of α. The generators of one system will cut t, those of the other not cutting π. There is just one tangent space at each point of c_2, hence Γ contains a $(1, 1)$ system of lines not lying in the generator planes.

19. The apparent contour from a point not on Γ,

$$\Gamma \equiv x_3 x_1^2 + x_4 x_1 x_2 + x_5 x_2^2 = 0,$$

say $(0, 1, 0, 0, 1)$, is

$$4(x_1 x_4 - 3x_2^2)^3 + 27\,[x_3 x_1^2 + x_1 x_2 x_4 - 2x_2^3]^2 = 0,$$

which contains x_1^2 as a factor. The other factor is the envelope of the plane

$$k^3 x_1 - 3k^2 x_2 + k x_4 + x_3 = 0,$$

the projection of a generating plane.

From a point on Γ, the apparent contour consists of π counted twice, the generating plane through the center of projection and the point which is the projection of the transversal through it.

20. In the preceding case, the tangent spaces formed an involution. Now suppose one factor x_3 is fixed while the other space turns about π projective with the point of tangency on d. In the equation of Γ, a, b, h, c can contain only x_1, x_2, x_3, while f, g are general. The equation of the variable tangent S_3 is

$$\xi_4 (g_4 x_1 + f_4 x_2) + \xi_5 (g_5 x_1 + f_5 x_2) = 0.$$

The fixed space $x_3 = 0$ cuts Γ in

$$(a_1 x_1 + a_2 x_2)\, x_1^2 + (h_1 x_1 + h_2 x_2)\, x_1 x_2 + (b_1 x_1 + b_2 x_2)\, x_2^2 = 0;$$

i. e., in three planes π_1, π_2, π_3 passing through d. The section made by any S_3 through π_i, say

$$\alpha_i x_1 + \beta_i x_2 = k x_3,$$

will be a quadric

$$\psi_2 (x_1, x_2) + (f_4 x_4 + f_5 x_5)\, x_2 + (g_4 x_4 + g_5 x_5)\, x_1 = 0,$$

whose section with π_i is

$$x_1 (B x_1 + (f_4 x_4 + f_5 x_5)\, \alpha_i + (g_4 x_4 + g_5 x_5)\, \beta_i) = 0,$$

thus the line d and a variable line, meeting d in the fixed point of contact of the space $\pi_i \pi$.

21. If Γ has a double point D not lying on d, the plane Dd will lie on Γ, hence D will either lie in π or in $\pi_i = \pi_k$.

Let $\pi_1 = \pi_3$ and P_1 be the point of contact on d of the space $\pi_1 \pi$. A space passing through π_1 will cut Γ in π_1 and a quadric, the section of the latter by π_1 being d and $P_1 D$; hence this line must be a double line. If π_1 be $x_1 = 0, x_3 = 0$, then $b_1 = b_2 = h_2 = 0$. The point on d associated with $x_1 = 0$ is $(0, 0, 0, f_5, -f_4)$, and the equations of the new double line are

$$x_1 = 0, \quad b_3 x_2 + f_2 x_2 + f_4 x_4 + f_5 x_5 = 0, \quad x_3 = 0.$$

This is the most general Γ having a double line of the second species and another double line of the first species. There are evidently three (1, 3) systems of lines, one cutting d, one cutting the other double line, and the third cutting π_2.

22. Now suppose $D \equiv (0, 0, \xi_3, \xi_4, \xi_5)$ in π. This requires that $c \equiv 0$, $g_3 = 0$, $f_3 = 0$.

$$\Gamma \equiv \prod_{i=1}^{3} (a_i x_1 + \beta_i x_2) + x_3 (f x_2 + g x_1) = 0.$$

The space $\pi \pi_i$ cuts Γ in $\sigma_i \equiv f \beta_i - g a_i = 0$, which passes through D, and cuts d in the point belonging to $\pi \pi_i$. Γ now has four systems (1, 2), of which the first cuts d, and the others cut π_i, σ_{i+1}, σ_{i+2}.

The general equation of Γ is

$$\Gamma \equiv x_1 x_2 (x_1 - x_2) + x_3 (x_1 x_4 + x_2 x_5) = 0.$$

$$\begin{array}{llll}
\pi: & x_1 = 0, \ x_2 = 0. & \sigma_1: & x_1 = 0, \ x_5 = 0. \\
\pi_1: & x_1 = 0, \ x_3 = 0. & \sigma_2: & x_2 = 0, \ x_4 = 0. \\
\pi_2: & x_2 = 0, \ x_3 = 0. & \sigma_3: & x_1 - x_2 = 0, \ x_4 + x_5 = 0. \\
\pi_3: & x_1 - x_2 = 0, \ x_3 = 0. & &
\end{array}$$

The equations of the four systems are

$$\begin{array}{ll}
\alpha x_2 + \beta x_1 = 0, & \alpha x_2 + \beta x_4 = 0, \\
\alpha x_4 + \gamma (x_2 - x_1) - \beta x_5 = 0, & \beta x_1 - \gamma x_3 = 0, \\
\beta x_1 - \gamma x_3 = 0, & \alpha x_1 - \beta x_5 + \gamma (x_2 - x_1) = 0, \\
\alpha x_1 + \beta x_5 = 0, & \alpha x_2 + \beta x_4 = 0, \\
\alpha x_2 - \beta x_4 + \gamma (x_2 - x_1) = 0, & \alpha x_1 - \beta x_5 + \gamma x_1 = 0, \\
\beta x_2 - \gamma x_3 = 0, & \beta (x_1 - x_2) + \gamma x_3 = 0.
\end{array}$$

23. A double line $(x_1 = 0, x_3 = 0, x_5 = 0)$ and a double point $(0, 0, 1, 0, 0)$ both lie on Γ, when its equation is of the form

$$\Gamma \equiv x_1^2 x_2 + x_3 (x_1 x_4 + x_2 x_5) = 0.$$

In the preceding table, π_3 coincides with π_1, and σ_3 with σ_1. There are therefore three systems of lines (1, 2), the first cutting d, the second d', σ_2, the third π_2, σ_1.

24. In 19–22 the fixed basis plane π did not lie in the fixed tangent space. Now let $x_1 = 0$ be the fixed tangent space, π as before. In this case Γ can have no other double point nor double line. It has four $(1, 3)$ systems.

25. The apparent contour from a point on Γ is of order 4, class 6, except when Γ has a double point, in which case it is of class 4. The general one is [3111], formed by the lines of a quadratic complex which meet a line belonging to the complex. If Γ has the double point and a double line we have [123], wherein d is a line of the complex and is tangent to the Kummer surface which is its surface of singularities. From $(1, 1, 1, -1, 0)$ on $x_1 = 0$ the equation is

$$[x_3 x_4 + x_3 x_5 + x_2 x_5]^2 - 4 [x_2 - x_3 + x_4 + x_5] x_2 x_3 x_5 = 0.$$

The line $x_2 = 0$, $x_3 = 0$ is the cuspidal line, $x_3 = 0$, $x_5 = 0$ the double one. The projection of D is $(0, 1, 0, 0)$. The plane $x_2 = 0$ cuts F_4 in the cuspidal line and touches it along the line $x_4 + x_5 = 0$, projection of σ_1. $x_5 = 0$ passes through the double line and touches the surface along $x_4 = 0$. The plane $x_2 - x_3 + x_4 + x_5 = 0$ touches it along the conic

$$(x_3 - x_5)(x_4 + x_5) + x_3 x_5 = 0.$$

The other double point is at $(0, 1, 1, -1)$.

If the center of projection be taken in π, the double line becomes tacnodal. The equation is

$$(x_1^2 + x_3 x_5)^2 - 41 x_3^2 x_4 = 0.$$

$(0, 0, 1, 0)$ is a triple point on the tacnodal line.

From a point not on Γ the contour F_6 is of class 6, has a cuspidal c_6 and a cuspidal line intersecting in a triple point of F_6. It is focal surface for four $(3, 3)$ congruences, one of which has the cuspidal line for focal line. If Γ has a double line, F_6 will have a double line intersecting c_6 in another triple point on the surface. One of the $(3, 3)$ congruences has the new double line for focal line. If Γ has a double point, F_6 will be of class 4. It has four systems $(3, 2)$. If Γ has both a double line and a double point, F_6 is of class 4, has a cuspidal line and an intersecting double line, and a double point. The cuspidal c_6 has two double points, d being tangent to a branch at one node, d' being tangent to a branch at the other.

If, however, the center of projection is in the plane π, the contour will be a ruled surface having a triple directrix, the three generators issuing from any

point lying in a plane containing the directrix. It has three cuspidal generators, and is contained in a special linear congruence.*

But if Γ has a double point, π lies on Γ, and the contour becomes a ruled quartic contained in a special linear congruence. From $(0, 0, 1, 1, 1)$ it becomes

$$(x_1 x_4 + x_2 x_5)^2 - 4(x_1^2 - x_2^2) x_1 x_2 = 0.$$

It has the symbol $[(21) 111]$.

If Γ has both D and d', the corresponding contour

$$(x_1 x_4 + x_2 x_5)^2 - 4(x_1 + x_2) x_1^2 x_2 = 0$$

has the double generator $x_1 = 0$, $x_5 = 0$. Symbol $[(12) 12]$. From $(0, 0, 1, 0, 0)$ we have an F_4 with a triple line. It has the symbol $[(31) 11]$.

§ 5. *Two Double Lines of Second Species.*

26. Let $d\,(x_1 = 0,\ x_2 = 0,\ x_3 = 0)$ and $d'\,(x_1 = 0,\ x_3 = 0,\ x_4 = 0)$ be two lines on Γ, each of the second species. Along d, $x_3 = 0$ is a fixed tangent space, the other one describing a pencil projective with the points of d, the basis plane being $x_1 = 0$, $x_2 = 0$. Similarly, $x_3 = 0$ is fixed for points of d', the basis plane of the variable tangent space being $x_1 = 0$, $x_4 = 0$. The equation may be written

$$(a_1 x_1 + a_3 x_3) x_1^2 + h_3 x_1 x_2 x_3 + g x_1 x_3 + (f_1 x_1 + f_4 x_4) x_2 x_3 + (c_1 x_1 + c_3 x_3) x_3^2 = 0.$$

The tangent space $x_3 = 0$ cuts Γ in the plane $x_1 = 0$ counted three times. The section of Γ by $x_1 = 0$ is

$$x_3 (f_4 x_2 x_4 + c_3 x_3^2) = 0,$$

i. e., the fixed plane containing the double lines and a quadric cone containing d, tangent plane $x_1 = 0$, $x_2 = 0$, and containing d', tangent plane $x_1 = 0$, $x_4 = 0$. If $c_3 = 0$, the cone breaks up into these two tangent planes. The space $x_2 = kx_1$ cuts Γ in a quadric whose section with $x_1 = 0$, $x_2 = 0$ consists of d and the variable line $g + k f_4 x_4 + c_1 x_3 = 0$ passing through the fixed point $x_4 = 0$, $g + c_1 x_3 = 0$. Let $g \equiv x_5$. To avoid duplication of terms we may write

$$a_1 = h_3 = f_4 = c_1 = 1, \qquad f_2 = 0.$$

* This is included in type IX of my enumeration of sextic scrolls of genus one, JOURNAL, Vol. XXV (1903), p. 87. If Γ has d', F_6 is of genus zero. It is now type XXVIII of rational surfaces. See p. 74.

Γ has an additional double point at $(0, 1, 0, 0, -1)$, and can be generated by the two conjugate systems

$$\begin{aligned}&\alpha x_2 - \beta x_1 = 0,\\ &\alpha x_1 - \gamma x_3 = 0,\\ &\alpha (x_3 + x_5) + \beta (x_1 + x_4) - \gamma (x_1 + a_3 x_3) = 0;\end{aligned}$$

$$\begin{aligned}&\alpha x_2 + \beta x_1 + \gamma (x_3 + x_5) = 0,\\ &-\alpha x_1 + \gamma (x_1 + x_4) = 0,\\ &\beta x_3 + \gamma (x_1 + a_3 x_3) = 0.\end{aligned}$$

The first system cuts d, the second cuts d'. Each is $(1, 2)$.

27. The apparent contour from a point on Γ is an F_4 with two coplanar cuspidal lines. When the double point exists, it becomes the Plücker complex surface of symbol [33]. If the center of projection be taken in the plane $x_1 = 0$, $x_3 = 0$, the result is a cubic with 4 double points, with symbol [222]. From any point in either basis plane the contour consists of two quadrics touching each other along the projections of the cuspidal lines. The symbol is [(21)(11)1].

From $(1, 0, 0, 0, 0)$ not on Γ, the contour becomes a double plane and a surface of order 4 having a cuspidal conic. If $a_3 = 0$, the cuspidal conic reduces to two straight lines intersecting in a triple point on the surface at $(0, 1, 0, -1)$. The equation is

$$4 x_3 (x_2 + x_3 + x_5)^3 + 27 x_2^2 x_4^2 = 0.$$

28. The two basis planes may lie in the same space. The equation may now be written in the form

$$\Gamma \equiv x_1^2 x_5 + \lambda x_1 x_2^2 + x_2^3 + x_1 x_3 x_4 = 0,$$

$x_1 = 0$, $x_3 = 0$ being the basis plane for the variable spaces belonging to points of d, and $x_1 = 0$, $x_4 = 0$ being the plane for d'. From any point in $x_1 = 0$, not on Γ, the apparent contour is a quartic surface with a cuspidal conic. From $(0, 1, 0, 0, 0)$ it becomes two quadrics having two generators in common. From an arbitrary point not in $x_1 = 0$ the contour is a surface of order 6 having three cuspidal conics passing through two common points, at which they touch the same planes, and two cuspidal lines, the latter passing through the points of intersection of the conics and lying in the tangent planes.

From $(1, 0, 0, 0, 1)$ the surface is

$$4 (x_3 x_4 - 3 x_1^2)^3 + 27 (\lambda x_1 x_2^2 - 2 x_1^3 + \lambda x_2^3 + x_1 x_3 x_4)^2 = 0.$$

The cuspidal lines are $x_2=0$, $x_3=0$; $x_2=0$, $x_4=0$. The three cuspidal conics are cut from the cone $x_3 x_4 - 3x_1^2 = 0$ by the planes $x_2 = kx_1$, wherein

$$\lambda k^3 (k+1) = -1.$$

If one of these conics be regarded as the absolute, a number of types of cyclides and other related surfaces can be obtained.

§ 6. *Depiction of* Γ *on* S_3.

29. Any cubic variety containing a plane can be birationally mapped on ordinary space. Let a plane π and a non-incident line d lie on Γ. From any point of S_4 can be drawn one line cutting d, π. Thus, any point P of Γ is uniquely associated with the point in S_3 in which the line through P cutting d, π cuts S_3. Since any six spaces in S_4 can be projected into any given ones not having an S_2 in common, we may define π by $x_1=0$, $x_2=0$; d by $x_3=0$, $x_4=0$, $x_5=0$ and S_3 by $x_1+x_2+x_3+x_4+x_5=0$. The equations of the line connecting $(\xi)=(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ on Γ with d, π are

$$\xi_2 x_1 - \xi_1 x_2 = 0, \quad \xi_4 x_3 - \xi_3 x_4 = 0, \quad \xi_3 x_5 - \xi_5 x_3 = 0,$$

and the corresponding point in S_3 becomes

$$\tau x_1 = -\xi_1(\xi_3+\xi_4+\xi_5), \qquad \tau x_2 = -\xi_2(\xi_3+\xi_4+\xi_5),$$
$$\tau x_3 = \xi_3(\xi_1+\xi_2), \qquad \tau x_4 = \xi_4(\xi_1+\xi_2), \qquad \tau x_5 = \xi_5(\xi_1+\xi_2).$$

The equation of Γ may be written in the form

$$\Gamma \equiv (a\xi_3 + b\xi_4 + c\xi_5)\xi_1 - (a'\xi_3 + b'\xi_4 + c'\xi_5)\xi_2,$$

in which $a = \sum_{i=1}^{5} a_i \xi_i$, and similarly for the other terms. By means of the equations of the line we can now solve for ξ_i, giving the following results:

$$\begin{aligned}
\sigma\xi_1 &= x_1[x_2\Sigma' - x_1\Sigma],\\
\sigma\xi_2 &= x_2[\qquad \text{“} \qquad],\\
\sigma\xi_3 &= x_3[A_1 x_1^2 + A_{12}x_1x_2 + A_2' x_2^2],\\
\sigma\xi_4 &= x_4[\qquad\qquad \text{“} \qquad\qquad],\\
\sigma\xi_5 &= x_5[\qquad\qquad \text{“} \qquad\qquad],
\end{aligned}$$

wherein

$$\begin{aligned}
A_i &= a_i x_3 + b_i x_4 + c_i x_5,\\
A_{12} &= (a_2 - a_1')x_3 + (b_2 - b_1')x_4 + (c_2 - c_1')x_5,\\
\Sigma &= A_3 x_3 + A_4 x_4 + A_5 x_5,
\end{aligned}$$

with similar expressions for A_i', Σ_i'.

Any line of Γ which cuts both π and d will go into a point; one cutting d and not π, or π and not d will go into a straight line; one cutting neither will go into a conic. The line d and the plane π are principal elements in the depiction. The lines cutting two transversals of d, π can be depicted on a system of circles with common absolute points.

A linear transformation which leaves Γ invariant will define a Cremona transformation in S_3, and any other variety left invariant by the same transformation will define an invariant surface in S_3. Conversely, corresponding to any linear or Cremona group in S_3 there will be a group of birational transformations leaving Γ invariant.

An interesting application can be made to the varieties given in Nos. 19, 22, 23, 28, when we consider the one-parameter groups

$$\sigma \xi_1 = \xi_1', \quad \sigma \xi_2 = \xi_2', \quad \sigma \xi_3 = \xi_3', \quad \sigma \xi_4 = \xi_4' - \lambda \xi_2', \quad \sigma \xi_5 = \xi_5' + \lambda \xi_1'$$

under which Γ remains invariant. Any function of $\xi_1, \xi_2, \xi_3, \xi_1 \xi_4 + \xi_2 \xi_5$, together with Γ, will define a surface in S_4 which will become a surface in S_3 invariant under a continuous Cremona group. But the resulting suface will contain only x_1, x_2, x_3 and hence defines a cone. From the theory of birational transformations of plane curves it now follows that the genus of this cone can not be greater than one, whatever function be chosen.

Another illustration is furnished by the variety

$$x_1^2 x_4 - x_2^2 x_5 + x_1 x_3 x_5 = 0,$$

a particular case of that considered in No. 19. It was shown by Fano* that it is invariant under the four-parameter continuous linear group defined by

$$\begin{aligned}
x_1' &= x_1, \\
x_2' &= \rho (x_2 + \alpha x_1), \\
x_3' &= \rho^2 (x_3 + 2 \alpha x_2 + (\alpha^2 - \beta) x_1), \\
x_4' &= \sigma \rho^2 (x_4 + \beta x_5), \\
x_5' &= \sigma x_5 .
\end{aligned}$$

* "Sulle varietà algebriche dello spazio a quattro dimensioni ....," *Atti Ist. Venuto*, Ser. 7, Vol. VII (1896), pp. 1069–1108.

The corresponding birational transformation in $\Sigma x_i = 0$ becomes

$$\begin{aligned}
\tau x_1' &= -x_1 A,\\
\tau x_2' &= -\rho\,(x_2 + \alpha x_1)\,A,\\
\tau x_3' &= \rho^2 B x_3\,(C - 2\alpha x_1 x_2 x_5 - (\alpha^2 - \beta)\,x_1^2 x_5),\\
\tau x_4' &= \sigma \rho^2 BC\,(x_4 + \beta x_5),\\
\tau x_5' &= \sigma x_5 BC.
\end{aligned}$$

$$\begin{aligned}
A &= \rho^2 x_3 C - 2\alpha \rho^2 x_1 x_2 x_3 x_5 - \rho^2(\alpha^2 - 1)\,x_1^2 x_3 x_5 + C\sigma\,(\rho^2 x_4 + (\rho^2 \beta + 1)\,x_5),\\
B &= \rho x_2 + (\rho\alpha + 1)\,x_2,\\
C &= x_1^2 x_4 - x_2^2 x_5.
\end{aligned}$$

The pencil of quadric varieties $x_2^2 - x_1 x_3 = \lambda x_1^2$ and the pencil of spaces $x_1 = \mu x_5$ remain invariant under this group. In particular the form

$$(1 - \rho^2)\,(\xi_1^2 - \xi_1 \xi_3) - \rho^2 \beta\, \xi_1^2$$

is multiplied by ρ^2, and $\xi_1 \xi_5$ is multiplied by σ. Various functions of these two forms will therefore preserve their form when an appropriate relation between ρ, σ is given. Thus, if $\sigma = \rho^2$, we obtain the quartic surface

$$x_1^2 x_3\,[(1 - \rho^2)\,x_4 - \rho^2 \beta x_5] + x_5\,(x_1^2 x_4 - x_2^2 x_5) = 0$$

which has the three-parameter group. It is of type [2211]. This surface can be obtained by the method of Cremona.* Another form of quartic is obtained when $\sigma \rho^2 = 1$. When $\sigma^3 = \rho^2$, we obtained the surface of order 10

$$x_1^6 x_3^3\,[(1 - \rho^2)\,x_4 - \rho^2 \beta x_5] = x_5\,(x_1^2 x_4 - x_2^2 x_5)^3.$$

All these surfaces can be mapped birationally upon an arbitrary plane.

CORNELL UNIVERSITY, *December*, 1908.

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* "Rappresentatione piana di alcune superficie algebriche dotate di curve cuspidali," *Bologna Mem.*, Ser. 3, Vol. II (1872). See also Cremona's note in the *Göttinger Nachrichten*, 1871.

Finite Groups Which May Be Defined by Two Operators Satisfying Two Conditions.

BY G. A. MILLER.

§ 1. *Introduction.*

In the present article the expression, "condition satisfied by two operators (s_1, s_2)," will be used exclusively for a relation which may be represented in the form $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \ldots . = 1$, where $\alpha, \beta, \gamma, \delta, \ldots .$ are positive or negative integers. Unless the contrary is explicitly stated, the operators s_1, s_2 are supposed to be distinct and neither of them is the identity. It is clear that a single condition between these operators is insufficient to define a finite group; for, if s_1, s_2 satisfy the equation $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \ldots . = 1$ but are not restricted in any other way, we may replace s_1, s_2 respectively by two commutative operators t_1, t_2 which have been so chosen that $t_1^{\alpha+\gamma+\ldots .}$ is the inverse of $t_2^{\beta+\delta+\ldots .}$. As this condition does not fix an upper limit for the orders of t_1, t_2, it results that the order of the group (G) generated by s_1, s_2 can not be limited by a single condition imposed upon s_1, s_2.

Since it is necessary to have at least two conditions between two or more operators to limit the order of the group or groups which may be generated by them, it is of interest to inquire what groups may be defined by the least possible number of conditions between two operators. Such a study appears especially useful in the application of group properties to other subjects.

The two conditions can not be of the form $s_1^\alpha = 1$, $s_2^\beta = 1$, since the order of the product of two operators is not limited by the orders of these operators. If the conditions are of the form

$$s_1^n = 1,^* \qquad s_1^\alpha = s_2^\beta,$$

the order of G has an upper limit when n and α are relatively prime, or when

* A defining equation of the form $s_1^n = 1$ implies that the order of s_1 is exactly n.

$\beta = 1$, and only then. When one of these conditions is satisfied, G is a cyclic group and its order divides βn. Moreover, every cyclic group can clearly be generated in this way. For instance, the cyclic group of order 40 is completely defined by the conditions $s_1^{10} = 1$, $s_1^8 = s_2^4$. On the contrary, the two conditions $s_1^{10} = 1$, $s_1^4 = s_2^8$ are satisfied by two operators which generate a group whose order exceeds any given number, which is in accord with the general theorem stated above.

It may happen that a pair of conditions which does not impose an upper limit on the order of G is nevertheless very useful in exhibiting properties of G. While such pairs of conditions do not lie within the scope of the present paper, it seems desirable to consider briefly one such pair, in view of its usefulness in what follows. This pair is

$$s_1^\alpha = s_2^\beta, \qquad s_1 s_2 = s_2 s_1.$$

As s_1, s_2 are commutative, G is an abelian group, and the following theorem is evident: *If two commutative operators satisfy the condition $s_1^\alpha = s_2^\beta$, they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of these operators, and a divisor of the highest common factor of α, β. The latter group may be the identity, and it must be the identity whenever α, β are relatively prime.*

Two operators which have a common square are known to generate a very elementary category of solvable groups.* By adding to this condition the order of their product there result the pair of conditions

$$(s_1 s_2)^n = 1, \qquad s_1^2 = s_2^2.$$

These are satisfied by the generators of only a finite number of groups for a given value of n, as may be seen from the following developments:

$$(s_1 s_2^{-1})^n = (s_1 s_2 \cdot s_2^{-2})^n = (s_1 s_2)^n \cdot s_2^{-2n} = s_2^{-2n},$$

since s_2^2 is invariant under G. As s_2^{-2n} is both invariant under G and also transformed into its inverse by each of the operators s_1, s_2, it results that the order of s_2 divides $4n$, and that the order of $s_1 s_2^{-1}$ divides $2n$. If these operators have the highest possible orders for a given value of n, the cyclic groups which they generate have two common operators; and hence it results that the order of G

* *Archiv der Mathematik und Physik*, Vol. IX (1905), p. 6.

is a divisor of $4n^2$. An interesting illustrative example is furnished by the non-abelian group of order 16 which contains two cyclic subgroups of order 8.*

$$s_1 = ahgfedcb \,.\, iponmlkj, \qquad s_2 = ajgpencl \,.\, bohmfkdi,$$
$$s_1 s_2 = am \,.\, bj \,.\, co \,.\, dl \,.\, ei \,.\, fn \,.\, gk \,.\, hp, \qquad s_1 s_2^{-1} = aoek \,.\, blfp \,.\, cigm \,.\, dnhj.$$

In this example $n = 2$ and the order of G is exactly $4n^2$. On the other hand, the two conditions

$$(s_1 s_2)^2 = 1, \qquad s_1^2 = s_2^2$$

can not be satisfied by the two generators of another non-abelian group. If G is abelian, it is either the four group or the group of order 8 and of type (2, 1). Hence the theorem: *If the two generators of a group satisfy the conditions* $(s_1 s_2)^2 = 1$, $s_1^2 = s_2^2$, *the group is one of the following three: the four group, the abelian group of order* 8 *and of type* (2, 1), *the non-abelian group of order* 16 *which contains two cyclic subgroups of order* 8.

In a similar manner all the possible groups may be determined for any given value of n. Since the second condition is sufficient to restrict G to a very elementary category of solvable groups, it seems of little value to determine all the possible groups for special values of n. The theorem that the number of these groups corresponding to a given value of n must be finite is, however, useful. A much more interesting set of conditions is

$$s_1^n = 1, \qquad s_1 s_2 = s_2^\alpha s_1^\beta.$$

When either α or $\beta = 0$, or when $\alpha = 1 = \beta$, these conditions reduce to those which were considered above. These special cases will be excluded in what follows, and we shall therefore confine ourselves to non-abelian groups in considering the groups whose generators satisfy such conditions. A large number of interesting cases arise when either α or $\beta = \pm 1$. These cases will be completely investigated, as regards the number of possible groups, in what follows. A number of these cases were considered recently under the heading "Generalization of the Groups of Genus Zero,"† and the present article completes this generalization along certain lines, especially when $\beta = -1$.

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* *Quarterly Journal of Mathematics*, Vol. XXVIII (1896), p. 269.

† *Transactions of the American Mathemati*[illegible], Vol. VIII (1907), p. 1.

§ 2. *Conditions of the Form* $s_1^n = 1,\ s_1 s_2 = s_2^\alpha s_1^\beta$, *when* $\alpha = \pm 1$ *or when* $\beta = 1$.

As observed above, it may be assumed that neither of the two numbers α, β is 0, and that they are not both equal to unity. If both of them are equal to -1, the second condition implies only that the order of $s_1 s_2$ is 2; and hence it does not limit the order of s_2. That is, the two conditions $s_1^n = 1,\ s_1 s_2 = s_2^{-1} s_1^{-1}$ are not sufficient to fix an upper limit for the order of G, and hence do not come within the scope of the present paper. In what follows we may therefore assume that α and β are not both equal to -1.

With the restrictions noted in the preceding paragraph, we proceed to determine all the possible groups when $\beta = 1$. Since $s_1 s_2 s_1^{-1} = s_2^\alpha$, it results that $s_1^n s_2 s_1^{-n} = s_2 = s_2^{\alpha^n}$. That is, the order of s_2 divides $\alpha^n - 1$ and is completely determined when this number is a prime, and only then. In this case G is completely defined by the given conditions, and in all other cases they restrict G to a limited number of groups. For instance, when $s_1^5 = 1,\ s_1 s_2 = s_2^2 s_1$, it results that s_2 is of order 31 and that G is the subgroup of order 155 in the holomorph of the group of order 31. On the other hand, the conditions $s_1^6 = 1,\ s_1 s_2 = s_2^2 s_1$ are satisfied by two generators of the group of order 378 obtained by extending the cyclic group of order 63 by means of an operator of order 6 transforming a generator of this cyclic group into its square, and also by two generators of each one of its four subgroups of orders 18, 42, 54 and 126 respectively. When $\alpha = 1$, the order of G has no upper limit; for, if s_1, s_2 satisfy the given conditions $s_1, t_2 s_2$ (t_2 being any operator which is commutative with s_2) must also satisfy these conditions, since the second condition takes the form $s_2^{-1} s_1 s_2 = s_1^\beta$. Hence the theorem: *The conditions* $s_1^n = 1,\ s_1 s_2 = s_2^\alpha s_1^\beta$ *are satisfied by the two generators of only a finite number of groups when* $\beta = 1$ *and* $\alpha \neq 1$, *but they are satisfied by two generators of each one of an infinite system of distinct groups when* $\alpha = 1$.

Having considered all the possible cases when at least one of the two numbers α, β is unity, we may assume, in what follows, that neither of these exponents is unity. When $\alpha = -1$, the conditions assume the form

$$s_1^n = 1, \qquad s_1 s_2 = s_2^{-1} s_1^\beta.$$

From the latter it results that $s_1 s_2 = s_2^{-1} s_1^{-1} \cdot s_1^{\beta+1}$, or $(s_1 s_2)^2 = s_1^{\beta+1}$. Since $s_1^{\beta+1}$ is a power of $s_1 s_2$ and commutative with s_1, it must also be commutative with s_2; and hence it is invariant under G. As it is assumed that $\beta \neq -1$, it results that *G contains at least one invariant operator besides the identity whenever* $\alpha = -1$. When n and $\beta + 1$ are relatively prime, G must be abelian, and hence the second

condition becomes $s_2^2 = s_1^{\beta-1}$. From this it results that the order of G is either n or $2n$ and that G is also cyclic whenever n and $\beta+1$ are relatively prime. When n is even, $\beta+1$ and $\beta-1$ are odd, and hence G is of order $2n$ in this case. That is, *G is completely determined by the two conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-1} s_1^\beta$ whenever n is even and also n and $\beta + 1$ are relatively prime. If these conditions are satisfied, G is the cyclic group of order $2n$.* For instance, G is the cyclic group of order 24 when $s_1^{12} = 1$ and $s_1 s_2 = s_2^{-1} s_1^6$, or when $s_1^{12} = 1$ and $s_1 s_2 = s_2^{-1} s_1^4$; but when $s_1^9 = 1$, $s_1 s_2 = s_2^{-1} s_1^4$, G is either the cyclic group of order 9 or the cyclic group of order 18.

When n and $\beta+1$ are not relatively prime, the order of G has no upper limit. This fact results from the following considerations: Since G is generated by $s_1 s_2$, s_1, the second of the given conditions may be replaced by $(s_1 s_2)^2 = s_1^{\beta+1}$. Hence it results that if $s_1 s_2$, s_1 satisfy these conditions, they are also satisfied by $t s_1 s_2$, $t_1 s_1$ (where t, t_1 are commutative with $s_1 s_2$, s_1); and t is of order 2, while the order of t_1 is a common factor of n and $\beta + 1$. As t, t_1 may be so selected that they satisfy these conditions and that also the order of $t t_1$ is an arbitrary number, the theorem under consideration has been proved. It has therefore been established that the conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-1} s_1^\beta$ can determine a finite group only when n and $\beta + 1$ are relatively prime. If, in addition to this, n is an even number, G is the cyclic group of order $2n$, while it is either this cyclic group or the cyclic group of order n when n is an odd number.

§ 3. *General Considerations as Regards Conditions of the Form $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^{-1}$.*

These conditions may be represented as follows:

$$s_1^n = 1, \qquad (s_1 s_2)^2 = s_2^{\alpha+1}$$

Hence $s_2^{\alpha+1}$ is invariant under G and $(s_1 s_2)^2 = (s_2$ … $s_2^{\alpha+1}$. Moreover, if s_1, s_2 satisfy these conditions and if t_1, t_2 are commu… h each of the operators s_1, s_2, and satisfy the additional condition $t_1^n =$ … it is possible to replace s_1, s_2 respectively by $t_1 s_1$, $t_2 s_2$ in th… nd α are such as to restrict the group $\{t_1, t_2\}$ ge… order, it is possible to select t_1, t_2 so as t… generate a group whose order exceeds any … $s_1^n =$ …

within the scope of the present paper it is necessary that n, $\alpha + 1$ have one of the following pairs of values: $2, k$; $3, \pm 3$; $3, \pm 4$; $3, \pm 5$; $4, \pm 3$; $5, \pm 3$; $3, -2$; $4, -2$; $5, -2$.

The first of these pairs does not require consideration, as the second condition would become $s_2^\alpha s_1$, since $s_1 = s_1^{-1}$ when s_1 is of order 2. We also excluded the cases when $\alpha = -2$, since these conditions would imply that s_1, s_2 are commutative and hence the resulting conditions could be written in a form which was completely considered in § 1. It will be found that each pair of conditions corresponding to the given separate sets of values of n, $\alpha + 1$ will give rise to a small number of groups of comparatively low order. These groups are closely related to the groups of genus zero. In fact, they have these groups for quotient groups, so that many of their fundamental properties may be directly deduced from those of the groups of genus zero.

§ 4. *Groups Defined by the Conditions* $s_1^3 = 1,\ s_1 s_2 = s_2^\alpha s_1^{-1}$.

The following pairs of conditions require consideration when $n = 3$:

$$s_1^3 = 1,\ s_1 s_2 = s_2^2 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^3 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^4 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-3} s_1^{-1};$$
$$s_1^3 = 1,\ s_1 s_2 = s_2^{-4} s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-5} s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-6} s_1^{-1}.$$

The first set of conditions may be expressed as follows:

$$s_1^3 = 1, \qquad (s_1 s_2)^2 = s_2^3.$$

These conditions have been considered in an equivalent but slightly different form,* and it has been proved that *there are just two non-abelian groups which may be generated by two operators satisfying the conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$. *These are the tetrahedral group and the group of order* 24 *which does not involve a subgroup of order* 12. *W*[illegible] s_1, s_2 *are commutative, the order of G is* 3.

The second set of c[illegible]ns is

$$[illegible] 1, \qquad (s_1 s_2)^3 = s_2^4.$$

To f[illegible]rder of s_2 we may proceed as follows: Since [illegible]e, the product of one of these operators into [illegible]rmed into its inverse by each one of them. [illegible]duct as to arrive at an invariant operator,

this operator is either the identity or it is of order 2. In the present case we have

$$(s_1 s_2 s_1^{-1} s_2^{-1})^3 = (s_2^3 s_1 s_2^{-1})^3 = s_2^3 s_1 s_2^2 s_1 s_2^2 s_1 s_2^{-1} = s_2^6 s_1^{-1} s_2^4 s_1^{-1} s_2^4 s_1^{-1} s_2^{-3} = s_2^{12}.$$

Hence the order of s_2 must be a divisor of 24. When this order is 2, G is the symmetric group of order 6; when it is 3, G is abelian; and when it is 4, G is the symmetric group of order 24 according to a well-known definition of this symmetric group. If the order of s_2 is 6, it is clear that $\{s_1, s_2^3\} \equiv$ the symmetric group of order 6, since $(s_1 s_2^3)^2 = 1$; hence $\{s_2^4, s_1, s_2^3\} \equiv G$ is the direct product of this symmetric group and the group of order 3 when the order of s_2 is 6.

If the order of s_2 is 8, the given conditions may be written in the form

$$t_1^3 = t_2^4, \qquad (t_1 t_2)^3 = 1, \qquad t_2^8 = 1.$$

It is known* that these conditions define a group of order 48 involving the group of order 24 which does not contain a subgroup of order 12. When the order of s_2 is 12, it is easy to see that $\{s_2^{-3}, s_1 s_2^{-4}\} \equiv$ the symmetric group of order 24, since $(s_2^{-3})^4 = 1$, $(s_1 s_2^{-4})^3 = 1$, $(s_2^{-3} s_1 s_2^{-4})^2 = (s_2^{-7} s_1)^2 = (s_2^5 s_1)^2 = s_2^8 (s_2 s_1)^2 = 1$. As G is generated by this symmetric group and s_2^4, it results that G is the direct product of the symmetric group of order 24 and the group of order 3 whenever the order of s_2 is 12. Finally, when the order of s_2 is 24, it results that $\{s_2^3, s_2^{-6} s_1 s_2^{-4}\} \equiv$ to the group of order 48 noted above; for

$$(s_2^{-3})^4 = (s_2^{-3} s_1 s_2^{-4})^2,$$

$$(s_2^{-6} s_1 s_2^{-4})^3 = (s_2^{-10} s_1)^3 = (s_2^{-2} s_1)^3 = s_2^{-2} s_1 s_2^{-2} s_1 s_2^{-2} s_1 = s_2 s_1^{-1} s_2^{-3} s_2^3 s_1^{-1} s_2^{-3} \cdot s_2^3 s_1^{-1} \cdot s_2^{-1} = 1.$$

By adjoining s_2^8 to this group of order 48 we clearly obtain G. As the group generated by s_2^8 is composed of invariant operators and has only the identity in common with $\{s_2^3, s_2^{-6} s_1 s_2^{-4}\}$, G is the direct product of the group of order 3 and the group of order 48 given above, whenever the order of s_2 is 24. Combining these results, we arrive at the theorem: *If two generators of a non-abelian group satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^3 s_1^{-1}$, it must be one of the following six groups: the symmetric group of order 6, the symmetric group of order 24, a group of order 48 involving the group of order 24 which does not contain a subgroup of order 12, or the direct product of the group of order 3 and one of these three groups. When G is abelian, it is the cyclic group of order 6 or the group of order 3. In the last case the two operators s_1, s_2 can, however, not be distinct.*

The third one of the given sets of conditions may be represented as follows:

$$s_1^3 = 1, \qquad (s_1 s_2)^2 = s_2^5.$$

To find an upper limit for the order of s_2 we may proceed in the following manner:

$$s_1 s_2 s_1^{-1} s_2^{-1} = s_2^4 s_1^{-2} s_2^{-1} = s_2^4 s_1 s_2^{-1},$$
$$(s_2^4 s_1 s_2^{-1})^2 = s_2^4 s_1 s_2^3 s_1 s_2^{-1} = s_2^8 s_1^{-1} s_2^6 s_1^{-1} s_2^{-2} = s_2^{13} s_1 s_2^4 s_1 s_2^{-2},$$
$$(s_2^4 s_1 s_2^{-1})^3 = s_2^{13} s_1 s_2^4 s_1 s_2^2 s_1 s_2^{-1} = s_2^{23} s_1^{-1} s_2^3 s_1^{-1} s_2^5 s_1^{-1} s_2^{-2} = s_2^{27} s_1^{-1} s_2^2 s_1 s_2^{-3} = s_2^{32} s_1^{-1} s_2 s_1^{-1} s_2^{-3},$$
$$(s_2^4 s_1 s_2^{-1})^5 = s_2^{32} s_1^{-1} s_2 s_1^{-1} s_2^{-3} . s_2^{13} s_1 s_2^4 s_1 s_2^{-2} = s_2^{45}.$$

As s_2^{45} is invariant under G and is also transformed into its inverse by $s_1 s_2$, it results that the order of s_2 divides 90. We may also assume that it is a multiple of 5, since we may confine ourselves to the consideration of non-abelian groups.

When s_2 is of order 5, G is the icosahedral group according to a familiar definition of this group. When this order is 10, the conditions may be expressed as follows: $(s_1 s_2)^4 = 1$, $s_2^5 = (s_1 s_2)^2$, $s_1^3 = 1$. Letting $t_1 = s_1 s_2$, $t_2 = s_2^{-1}$, these conditions become $t_1^4 = 1$, $t_2^5 = t_1^2$, $(t_1 t_2)^3 = 1$. The latter conditions define a group known as G_{120},* and hence this is the group defined by $s_1^3 = 1$, $s_1 s_2 = s_2^4 s_1^{-1}$, when the order of s_2 is 10. When this order is 15, the icosahedral group is generated by s_1, s_2^6, since $(s_2^6)^5 = 1$, $s_1^3 = 1$, $(s_1 s_2^6)^2 = s_2^{10} (s_1 s_2)^2 = s_2^{15} = 1$. Hence it results that G is the direct product of the icosahedral group and the group of order 3, when s_2 is of order 15. If this order is 30, it is evident that G_{120} is generated by $s_1 s_2^6$, s_2^{-9}, since $(s_1 s_2^6)^4 = 1$, $(s_2^{-9})^5 = (s_1 s_2^6)^2$, $(s_1 s_2^{-3})^3 = s_1 s_2^{-3} s_1 s_2^{-3} s_1 s_2^{-3}$ $= s_1 s_2 s_1^{-1} s_2^{-4} . s_2^4 s_1^{-1} s_2^{-4} = 1$. Hence G is the direct product of G_{120} and the group of order 3, when the order of s_2 is 30. The next possible value of this order is 45. When s_2 is of order 45, $\{s_2^9, s_1 s_2^{12}\} \equiv$ icosahedral group, since $(s_2^9)^5 = 1$, $(s_1 s_2^{21})^2 = (s_1 s_2)^2 s_2^{40} = 1$, $(s_1 s_2^{12})^3 = (s_1 s_2^{-3})^3 = 1$. In this case G is therefore the direct product of the cyclic group of order 9, generated by s_2^5, and the icosahedral group. Finally, when the order of s_2 is 90, G_{120} is generated by $s_1 s_2^{21}$, s_2^9, since $(s_1 s_2^{21})^4 = 1$, $(s_2^9)^5 = (s_1 s_2^{21})^2$, $(s_1 s_2^{30})^3 = 1$. Hence G is the direct product of the cyclic group of order 9 and G_{120} when s_2 is of order 90. These results give rise to the following theorem: *Two non-commutative operators satisfying the conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^4 s_1^{-1}$ *must generate one of the following six groups: the icosahedral*

group, G_{120}, or the direct product of one of these groups and one of the two cyclic groups of orders 3 and 9 respectively. Each of these groups is completely determined by adding the order of s_2 to the given conditions, and hence it is completely defined by three conditions. If two commutative operators satisfy the conditions $s_1^3=1$, $s_1 s_2 = s_2^4 s_1^{-1}$, they generate the cyclic group of order 9.

The set of conditions $s_1^3=1$, $s_1 s_2 = s_2^{-2} s_1^{-1}$ implies that s_2 is invariant under G and hence G is abelian. As $(s_1 s_2)^2 = s_1^2 s_2^2 = s_2^{-1}$, it results that s_2^3 is of order 3, and hence *the conditions $s_1^3=1$, $s_1 s_2 = s_2^{-2} s_1^{-1}$ define the cyclic group of order 9.* The next set of conditions is equivalent to $s_1^3=1$, $(s_1 s_2)^2 = s_2^{-2}$. Since $s_1 s_2$ and s_2^{-1} have a common square, $s_1 s_2^2$ is transformed into its inverse by each of these operators. An upper limit for the order of s_2 results from $(s_1 s_2^2)^3 = s_2^6$, s_2^2 being invariant under G. As s_2^6 both is invariant under G and is also transformed into its inverse by $s_1 s_2$, its order can not exceed 2. From this it results that the order of s_2 divides 12 and that the order of $s_1 s_2^2$ divides 6. If both of these operators have the largest possible orders, the order of G is 36. The remaining properties of this group of order 36 may easily be deduced from the fact that $s_1 s_2$ may be represented by the following substitutions:

$$s_1 = ace \,.\, bdf \,.\, klm, \qquad s_2 = afedcb \,.\, ghij \,.\, km.$$

When s_2 is of order 6 and $s_1 s_2^2$ is of order 3, G is of order 18 and may be generated by

$$s_1 = ace \,.\, bdf \,.\, klm, \qquad s_2 = afedcb \,.\, km.$$

If $s_1 s_2^2$ is the identity when s_2 is of order 6, G is clearly the cyclic group of order 6, and if $s_1 s_2^2$ is of of order 2 when s_2 is of order 12, G is the cyclic group of order 12. When s_2 is of order 4, $s_1 s_2^2$ must be of order 6, and hence G is the associate-dihedral* group of order 12. When s_2 is of order 3, G is the group of order 3, and when s_2 is of order 2, G is the symmetric group of order 6. Uniting these results we arrive at the theorem: *There are exactly seven groups which may be generated by two operators satisfying the two conditions $s_1^3=1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$; these are the group of order 3, the two groups of order 6, the cyclic group and the associate-dihedral group of order 12, a group of order 18, and one of order 36.* This theorem illustrates the general theorem of § 1 as regards groups generated by two operators having a common square.

The remaining three sets of conditions are respectively equivalent to the following:

$$s_1^3=1,\ (s_1s_2)^2=(s_2^{-1})^3;\quad s_1^3=1,\ (s_1s_2)^2=(s_2^{-1})^4;\quad s_1^3=1,\ (s_1s_2)^2=(s_2^{-1})^5.$$

The possible groups that may be generated by two operators satisfying these conditions were determined in Vol. VIII of the *Transactions of the American Mathematical Society*, pp. 4, 6 and 11 respectively. Hence all the cases which may give rise to only a finite number of groups when $n=3$ have been considered.

§ 5. *Groups Defined by the Conditions* $s_1^4=1,\ s_1s_2=s_2^a s_1^{-1}$.

When $n=4$, the following three sets require consideration:

$$s_1^4=1,\ s_1s_2=s_2^2s_1^{-1};\quad s_1^4=1,\ s_1s_2=s_2^{-3}s_1^{-1};\quad s_1^4=1,\ s_1s_2=s_2^{-4}s_1^{-1}.$$

The first set can be satisfied only when the order of s_2 divides 12, since

$$(s_1s_2s_1^{-1}s_2^{-1})^3=(s_2^2s_1^2s_2^{-1})^3=s_2^2s_1^2s_2s_1^2s_2s_1^2s_2^{-1}=s_2^2s_1\,.\,s_1s_2\,.\,s_1\,.\,s_1s_2\,.\,s_1^2s_2^{-1}=s_2^6.$$

As the order of s_2 is a multiple of 3 when G is non-abelian, it is necessary to consider the cases which arise when the order of s_2 has one of the following values: 3, 6, 12. In the first case G is the symmetric group of order 24. When s_2 is of order 6, $\{s_1^2s_2,\ s_2^2\}\equiv$ the tetrahedral group, since

$$(s_2^2)^3=1,\ (s_1^2s_2^3)^2=1,\ (s_1^2s_2)^3=s_1^2s_2s_1^2s_2s_1^2s_2=s_1\,.\,s_1s_2\,.\,s_1\,.\,s_1s_2\,.\,s_1^2s_2=s_1s_2^4s_1s_2=s_2^6=1.$$

This group is invariant under G, since it involves

$$s_2^{-1}s_1^2s_2^3=s_2^5s_1^2s_2^3=s_2^3s_1^2s_2^5=s_2^2\,.\,s_1^2s_2^3\,.\,s_2^3,$$

$$s_1^{-1}s_2^2s_1=s_1^2\,.\,s_1s_2\,.\,s_2s_1=s_1^2s_2^2s_1^2s_2^2=s_2^3s_1^2\,.\,s_2^3\,.\,s_1^2s_2^3\,.\,s_2^{-4},\ \text{and}\ s_1^{-1}\,.\,s_1^2s_2\,.\,s_1=s_1s_2s_1=s_2^2.$$

By adjoining s_2^3 to this invariant tetrahedral group there results the direct product of the group of order 2 and the tetrahedral group. This direct product is also invariant under G, since s_2^3 is invariant under G. The group of order 24 just obtained involves s_2 and s_1^2. Hence G is the group of order 48 obtained by adjoining s_1 to this group of order 24. When the order of s_2 is 12, G involves the quaternion group as an invariant subgroup. This subgroup may be generated by $s_1^2s_2^3$, $s_2^{-1}\,.\,s_1^2s_2^3\,.\,s_2=s_2^{-1}s_1^2s_2^4$. The fact that these two operators generate the quaternion group results from the equations

$$(s_1^2s_2^3)^4=1,\quad (s_1^2s_2^3)^2=(s_2^{-1}s_1^2s_2^4)^2,$$

$$s_1^2\,.\,s_2^{-1}s_1^2s_2^4\,.\,s_1^2=s_2^3s_1^2s_2^{-1}s_1^2s_2s_1^2=s_2^3s_1^2s_2^{-1}s_1s_2^2s_1=s_2^3s_1^2s_2s_1^{-1}\,.\,s_2s_1=s_2^3s_1^2s_2s_1^{-2}s_2^2$$

This quaternion group is invariant under s_2, since

$$s_2^{-2}\, s_1^2\, s_2^5 = s_2\, s_1^2\, s_2^2,$$

and

$$s_1^2 s_2^3 \,.\, s_2^{-1} s_1^2 s_2^4 = s_1^2 s_2^2 s_1^2 s_2^4 = s_1^2 \,.\, s_2^2 s_1^{-1} \,.\, s_1^3 s_2^4 = s_1^3 s_2 s_1^3 s_2^4 = s_1^3 s_2^3 \,.\, s_2^{-1} s_1^{-1} \,.\, s_2^4$$
$$= s_1^3 s_2^2 s_1 s_2^3 = s_1^3 \,.\, s_2^2 s_1^{-1} \,.\, s_1^2 s_2^2 = s_2 s_1^2 s_2^3.$$

Hence $\{s_2^4,\ s_1^2 s_2^3,\ s_2^{-1} s_1^2 s_2^4\} \equiv$ the group of order 24 containing no subgroup of order 12. By adjoining the invariant operator s_2^8 to this G_{24}, we obtain a G_{48} whose properties are evident from these conditions. The given G_{24} is invariant under s_1, since

$$s_1 s_2^8 s_1 = s_1^2 s_2^8, \quad s_1^{-1} s_2^{-1} s_1^2 s_2^4 s_1 = s_1^{-1} s_2^2 s_1^2 s_2 s_1 = s_1^{-1} \,.\, s_2^2 s_1^{-1} \,.\, s_1^2 \,.\, s_1 s_2 \,.\, s_1 = s_2 s_1^2 s_2^3,$$
$$s_1^{-1} s_2^4 s_1 = s_1^2 s_2^3 \,.\, s_2^{-4} \,.\, s_2^6 = s_1^3 s_2^{-1} s_1^{-1} s_1 s_2^6 = s_1^3 s_2^{-2} s_1 s_2^6 = s_1^3 s_2^4 s_1.$$

The first two of these equations also complete the proof of the fact that the given quaternion group is invariant under G. As G_{24} is invariant under s_1, G_{48} must also have this property, and the latter contains s_1^2. Hence G is the group of order 96 obtained by adjoining s_1 to the given G_{48}. The quotient group of G with respect to G_{24} is cyclic. These results may be expressed as follows: *If two non-commutative operators satisfy the conditions $s_1^4 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$, they generate one of three groups of order* 24, 48 *and* 96 *respectively. The first of these is symmetric, and the other two have this group as a quotient group with respect to the invariant operators generated by s_2^8. If two commutative operators satisfy these conditions, they generate the cyclic group of order* 4.

The second and third sets of conditions are equivalent respectively to $s_1^4 = 1$, $(s_1 s_2)^2 = (s_2^{-1})^2$; $s_1^4 = 1$, $(s_1 s_2)^3 = (s_2^{-1})^3$. The latter of these has been investigated.* The former implies that the order of s_2 divides 16, since $(s_1 s_2^2)^4 = s_2^8$, s_2 being invariant under G. If s_2 is of order 16, the cyclic group of order 8 generated by $s_1 s_2^2$ has two operators in common with the cyclic group of order 16 generated by s_2, and each operator of the former is transformed into its inverse by a generator of the latter. Hence G is of order 64 when s_2 is of order 16, which is in accord with the general theorem stated in § 1, as $n = 4$. When s_2 is of order 8 and $s_1 s_2^2$ is of order 4, the cyclic groups generated by these operators have only the identity in common, since $(s_1 s_2^2)^2 = s_2^4$ implies that s_1 is of order 2. In this case G is therefore of order 32. If s_2 is of order 4 and $s_1 s_2^2$

is of the same order, G is the quaternion group when these operators have a common square, otherwise the order of G is 16. If the order of $s_1 s_2^2$ is 2, G is evidently abelian. Hence we have proved the theorem: *There are exactly four non-abelian groups which may be generated by two operators satisfying the conditions* $s_1^4 = 1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$. *The orders of these groups are* 8, 16, 32 *and* 64 *respectively. The only abelian groups whose generators satisfy these conditions are the cyclic group of order* 8, *and the group of order* 16 *and of type* (3, 1).

§ 6. *Groups Defined by the Conditions* $s_1^5 = 1,\ s_1 s_2 = s_2^{\alpha} s_1^{-1}$.

The following three sets are the only ones which require consideration:

$$s_1^5 = 1,\ s_1 s_2 = s_2^2 s_1^{-1};\quad s_1^5 = 1,\ s_1 s_2 = s_2^{-3} s_1^{-1};\quad s_1^5 = 1,\ s_1 s_2 = s_2^{-4} s_1^{-1}.$$

If the first set is satisfied, the order of s_2 divides 30, since

$$(s_1 s_2 s_1^{-1} s_2^{-1})^2 = (s_2^2 s_1^{-2} s_2^{-1})^2 = s_2^2 s_1^{-2} s_2 s_1^{-2} s_2^{-1} = s_2^2 s_1^3 s_2^2 s_1^3 s_2^{-1},$$

$$(s_2^2 s_1^{-2} s_2^{-1})^3 = s_2^2 s_1^3 s_2^2 s_1^3 s_2 s_1^{-2} s_2^{-1} = s_2^3 s_1^2 s_2^2 s_1 s_2^3 s_1^2 s_2^{-1} = s_2^5 s_1^3 s_2 s_1^{-1} s_2 s_1 s_2^3 s_1^{-1} s_2^{-2}$$
$$= s_2^8 s_1^3 s_2 s_1^3 s_2 s_1^{-1} s_2^{-2} = s_2^8 s_1^2 s_2 s_1^2 s_2^2 s_1^3 s_2^{-2},$$

$$(s_2^2 s_1^{-2} s_2^{-1})^5 = s_2^8 s_1^3 s_2 s_1^2 s_2^4 s_1^2 s_2^{-1} = s_2^{11} s_1 s_2^3 s_2^2 s_1 s_2^{-1} = s_2^{14} s_1 s_2 s_1 s_2^{-1} = s_2^{15}.$$

As the order of s_2 is a multiple of 3, when G is non-abelian, and a divisor of 30, it has one of the following values: 3, 6, 15, 30. If this value is 3, G is the icosahedral group. If the order of s_2 is 6, we can readily prove that G is a group of order 120, known as G_{120},* by letting

$$t_1 = s_1 s_2, \qquad t_2 = s_1^{-1} s_2^3,$$

and observing that

$$t_1^4 = 1, \qquad t_2^5 = t_1^2, \qquad (t_1 t_2)^3 = (t_2 t_1)^3 = s_2^{12} = 1.$$

When the order of s_2 is 15, it may be observed that

$$(s_1 s_2^{-3})^5 = 1, \qquad (s_2^{10})^3 = 1, \qquad (s_1 s_2^7)^2 = (s_1 s_2)^2 s_2^{12} = 1.$$

Hence $\{s_1 s_2^{-3}, s_2^{10}\} \equiv$ the icosahedral group and G is the direct product of the cyclic group of order 5 and the icosahedral group. Finally, when the order of s_2 is 30, it is evident that

$$(s_1 s_2^7)^4 = 1, \qquad (s_1 s_2^7)^2 = (s_2^3 s_1^{-1})^5 = s_2^{15}, \qquad (s_2^3 s_1^{-1} . s_1 s_2^7)^3 = 1;$$

and hence $\{s_1 s_2^7,\ s_2^3 s_1^{-1}\} \equiv G_{120}$. Moreover, $\{s_1 s_2^7,\ s_2^3 s_1^{-1},\ s_2^6\} \equiv G$, since it involves $s_1 s_2 = s_2^3 s_1^{-1}$, and hence also $s_2^3 s_1^{-1} . s_1 s_2^{-3} = s_2^{-1}$. From this it results that G is the direct product of the cyclic group of order 5 and G_{120} when the order of s_2 is 30. This proves the following theorem: *If two non-commutative operators satisfy the conditions $s_1^5 = 1$, $s_1 s_2 = s_2^3 s_1^{-1}$, they generate one of the following four groups: the icosahedral group, G_{120}, the direct product of one of these groups and the cyclic group of order 5. If two commutative operators satisfy these conditions, they generate the cyclic group of order 5.*

As the third of the three given pairs of conditions is equivalent to $t_1^2 = t_2^3$, $(t_1 t_2)^5 = 1$, the groups which may be generated by two operators satisfying these conditions are known.* It remains therefore only to consider the possible groups when

$$s_1^5 = 1, \qquad (s_1 s_2)^2 = s_2^{-3}.$$

In this case the order of s_2 divides 20, since $(s_1 s_2^2)^5 = s_2^{10}$, and hence s_2^{10} is both invariant under $s_1 s_2$ and also tranformed into its inverse by this operator. If the order of s_2 is 20, the order of $s_1 s_2^2$ is 10, since G is not abelian. The order of G is therefore 100 when s_2 is of order 20. If the order of s_2 is 10, that of $s_1 s_2^2$ must be 5, and G is the group of order 50, which may be obtained by extending the group of order 25 and of type (1, 1) by means of an operator of order 2 which is commutative with one of the two independent generators of this group, but transforms the other into its inverse. If the order of s_2 is 4, that of $s_1 s_2^2$ is 10, and G is the associate-dihedral group of order 20, while G is the dihedral group of order 10 when s_2 is of order 2. As we have considered all the possible values of the order of s_2 that may give rise to a non-abelian group, there results the theorem: *There are exactly four non-abelian groups which may be generated by two operators satisfying the conditions $s_1^5 = 1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$. The orders of these groups are 100, 50, 20, 10 respectively. The abelian groups which may be generated by two such operators are cyclic and of orders 5, 10 and 20 respectively.*

§ 7. *Groups Defined by the Conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$; $s_1^n = 1$, $s_1 s_2 = s_2^2 s_1^2$.*

If two operators satisfy the condition $s_1 s_2 = s_2^{-2} s_1^{-2}$, they are either of the same order or the order of one of them is three times that of the other.

* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 12.

Moreover, the operator $s_1 s_2$ generates a cyclic group of odd order which is invariant under G, and the following relations exist for all values of m:

$$(s_1 s_2)^{\frac{2}{3} 4^m + \frac{1}{3}} = s_1^{2m+1} s_2^{2m+1}, \qquad (s_2 s_1)^{\frac{2}{3}(4^m - 1)} = s_1^{2m} s_2^{2m}.*$$

From these conditions it results that the order of G is finite for any finite value of n in the conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$. As G is also a solvable group of a very elementary type, it is easy to determine the possible groups for a given value of n. For instance, when $n = 2$, G is the cyclic group of order 6 or the group of order 2; when $n = 3$, G can be non-abelian only when it is of order 27 and involves operators of order 9. If G is abelian when $n = 3$, it is either of order 3 or it is the non-cyclic group of order 9. Hence the theorem: *If two generators of a group satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$, the group is one of the following three: the non-abelian group of order 27 which contains operators of order 9, the non-cyclic group of order 9, or the group of order 3.*

The merits of this theorem consist mostly in serving as a type of theorems which may be established for the different values of n. The fact that deserves emphasis in this connection is that for every given value of n the number of the possible groups is finite, and that these groups have such elementary properties as to make a complete determination of the possible groups for a given value of n an easy matter. The conditions

$$s_1^n = 1, \qquad s_1 s_2 = s_2^2 s_1^2$$

present much greater difficulties. These have been solved only for a few values of n. The latter condition implies that the orders of s_1, s_2 are either the same or that the order of one of these operators is odd and that of the other is the double of this odd number. When $n = 3$, G is the alternating group of order 12, the group of order 24 which does not contain a subgroup of order 12, or the group of order 3. When $n = 4$, G is either the cyclic group of order 4 or the holomorph of the group of order 5.†

§ 8. *Conclusion.*

The pairs of conditions which have been completely investigated, in the present paper, as regards whether they may be satisfied by two generators of

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* *Bulletin of the American Mathematical Society*, Vol. XV (1909), p. 163.

† Netto, *Crelle*, Vol. CXXVIII (1905), p. 255.

Symmetric Binary Forms and Involutions.

BY ARTHUR B. COBLE.

§ 1. *Introduction.*

An involution determined by $k+1$ linearly independent binary forms of order n,

$$(\alpha_1 x)^n,\ (\alpha_2 x)^n,\ \ldots,\ (\alpha_{k+1} x)^n,$$

is an aggregate of sets of n points in the binary domain, any k of which may be chosen at random, the remaining $n-k$ being then uniquely determined. We shall say that such an involution is of *order* n, of *freedom* k, and of *extent* $n-k$; and indicate it by the symbol $I_{k,\,n-k}$.*

According to Gordan,† every combinant of the given $k+1$ forms is a comitant of the so-called "fundamental combinant"

$$F \equiv \begin{vmatrix} (\alpha_1 x_1)^n & (\alpha_2 x_1)^n & \ldots & (\alpha_{k+1} x_1)^n \\ (\alpha_1 x_2)^n & (\alpha_2 x_2)^n & \ldots & (\alpha_{k+1} x_2)^n \\ \ldots & \ldots & \ldots & \ldots \\ (\alpha_1 x_{k+1})^n & (\alpha_2 x_{k+1})^n & \ldots & (\alpha_{k+1} x_{k+1})^n \end{vmatrix},$$

a form in $k+1$ variables, $x_1, x_2, \ldots, x_{k+1}$. This theorem is fairly evident from the interpretation of F. For, on the one hand, a combinant is by definition a comitant of the involution itself, rather than of the particular forms which determine it. On the other hand, the form F, equated to zero, is an analytic statement in invariant (*i. e.*, symbolic) form of the given involution. For it is the condition that the $k+1$ points $x_1, x_2, \ldots, x_{k+1}$ belong to a set of the involution; or if the k points $x_1, x_2, \ldots, x_k$ be given, it is the equation of order n satisfied by the k given values and the $n-k$ which they determine in the involution.

* Rather than by the usual unsymmetric symbol I^n_{n-k}.

† *Math. Annalen*, Vol. V.

Since F vanishes when x_i is x_j, where $i, j = 1, 2, \ldots, k+1$, and $i \neq j$, it contains every factor of the type $(x_i x_j)$. The other factor can be expressed symbolically as

$$H_{k,n-k} \equiv (a_1 x_1)^{n-k} (a_2 x_2)^{n-k} (a_3 x_3)^{n-k} \ldots (a_{k+1} x_{k+1})^{n-k},$$

the symbols $a_1, \ldots, a_{k+1}$ having a meaning only when all are combined to the degree $n-k$. In the identity, $F \equiv H_{k,n-k} \cdot \Pi(x_i x_j)$, F and Π each merely change in sign when x_i and x_j are interchanged. Hence $H_{k,n-k}$ is a symmetric form in the $k+1$ sets of variables.

The symmetric form $H_{k,n-k}$ thus determined by and determining an $I_{k,n-k}$ is not the most general form of its type. It will not usually happen that, when k points $x_1, x_2, \ldots, x_k$ are given and $n-k$ points determined by the equation $H_{k,n-k} = 0$ of degree $n-k$ in x_{k+1}, of the n points any set of k points will determine the remaining $n-k$ points. If, however, such a set of n points occurs for a given symmetric $H_{k,n-k}$, it will be called an *involutive set* of $H_{k,n-k}$. Evidently the extreme cases are, (a) the form $H_{k,n-k}$ has no involutive sets; (b) the form $H_{k,n-k}$ has ∞^k involutive sets. In the latter case, the involutive sets constitute an $I_{k,n-k}$, and the special form $H_{k,n-k}$ will also be denoted by $I_{k,n-k}$.

In many problems of projective geometry it is desirable to know how many involutive sets the given form $H_{k,n-k}$ possesses, and to find the conditions on the form that this number should increase until it attains its maximum, ∞^k. These questions are answered in part in this article and the results are interpreted geometrically. Two distinct geometrical representations of the form $H_{k,n-k}$ will be given first.

§ 2. *The Apolarity* Spread of $H_{k,n-k}$.*

Let

(1) $(cx)^m = (c_1 x_2 - c_2 x_1)^m = (-1)^m [c_2^m x_1^m - \binom{m}{1} c_2^{m-1} c_1 x_1^{m-1} x_2 + \ldots]$

represent symbolically the binary form of order m

(2) $(-1)^m [c_0 x_1^m - \binom{m}{1} c_1 x_1^{m-1} x_2 + \binom{m}{2} c_2 x_1^{m-2} x_2^2 - \binom{m}{3} c_3 x_1^{m-3} x_2^3 \ldots]$.

* A conspicuous example of the representation here employed is contained in the memoirs of Study, *Math. Annalen*, Vols. XXVII and XL. There the ternary quadratic is taken as the element, rather than the binary m-ic as above. Much of the content of this paragraph is implied in Study's Ternäre Formen, where (p. x) also there is found the notation for binary forms.

By the equations

(3) $X_0 = c_0$; $X_1 = -\binom{m}{1}c_1, \ldots, X_i = (-1)^i\binom{m}{i}c_{m-i}, \ldots$

the form is represented by a point X in S_m and conversely. All m-ics, $(cx)^m$, apolar to a given m-ic, $(dx)^m$, satisfy the linear condition, $(cd)^m = 0$, or, from (3),

(4) $X_0 d_m + X_1 d_{m-1} + X_2 d_{m-2} + \ldots + X_m d_0 = 0$;

i. e., the m-ics, $(cx)^m$, are represented by points X on an S_{m-1}, U, in S_m, whose coordinates are

(5) $U_0 = d_m, \; U_1 = d_{m-1}, \ldots, U_m = d_0.$

Thus the m-ic, $(dx)^m$, can also be represented by an S_{m-1}, U, and conversely. Wherever it is necessary to avoid confusion, we shall suppose that U represents $(cx)^m$ and X represents $(c\bar{x})^m$. The incidence condition for point X and space U is the apolarity condition for the corresponding binary forms.

To perfect m-th powers, $(y\bar{x})^m$, correspond points on the fundamental norm-curve, N_m, whose parametric equation in point form is

(6) $X_0 = y_2^m, \; X_1 = -\binom{m}{1}y_2^{m-1}y_1, \; X_2 = \binom{m}{2}y_2^{m-2}y_1^2, \ldots.$

The parametric equation of the norm-curve in reciprocal form, N_m, is

(7) $U_0 = y_1^m, \; U_1 = y_1^{m-1}y_2, \; U_2 = y_1^{m-2}y_2^2, \ldots.$

The space U of N_m osculates N_m at a point X with the same parameter. The equation $(cx)^m = 0$ determines the parameters of the m points X where the space $(cx)^m$ cuts N_m and also the parameters of the m spaces U of N_m which pass through the point $(c\bar{x})^m$.

Just as the form $(cx)^m$ is represented by an S_{m-1}, U, the locus of points X which represent forms $(d\bar{x})^m$ apolar to $(cx)^m$, so the symmetric form

(8) $H_{1,m} \equiv (a_1x_1)^m(a_2x_2)^m = (a_1x_2)^m(a_2x_1)^m$

is represented by a point-quadric. The points $(c\bar{x})^m$ and $(d\bar{x})^m$ are an apolar point-pair of the quadric if $(a_1c)^m(a_2d)^m = 0$; and the point $(c\bar{x})^m \equiv (c'\bar{x})^m$ is on the quadric if $(a_1c)^m(a_2c')^m = 0$. In particular, if x_1 and x_2 are values that satisfy $H_{1,m} = 0$, they are the parameters of two points on N_m, apolar to the quadric. The equation $(a_1x)^m(a_2x)^m = 0$ gives the parameters of the $2m$ points in which the quadric cuts N_m.

The same symmetric form can be considered dually to represent an S_{m-1}-quadric and will then be written with dashed variables. Evidently the

point-quadric $(a_1x_1)^m(a_2x_2)^m$ and the S_{m-1}-quadric $(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m$ are apolar if $(a_1\alpha_1)^m(a_2\alpha_2)^m = 0$.

The quadrics determined by the special symmetric forms

$$Q_i = (x_1x_2)^{2i}(a_1x_1)^{m-2i}(a_2x_2)^{m-2i} \qquad 0 < 2i \leqq m$$

are important. To characterize them we observe that the condition

$$(cc')^{2i}(a_1c)^{m-2i}(a_2c')^{m-2i} = 0$$

is satisfied when

$$(cx)^m = (c'x)^m = (s_1x) \cdot (s_2x) \cdot \ldots\ldots \cdot (s_{i-1}x) \cdot (tx)^{m-i+1}$$

for all values of $s_1, s_2, \ldots, s_{i-1}$, and t; and that when

$$(cx)^m = (c'x)^m = (s_1x) \cdot (s_2x) \cdot \ldots\ldots \cdot (s_ix) \cdot (tx)^{m-i},$$

the condition reduces to

$$(s_1t)^2 \cdot (s_2t)^2 \cdot \ldots\ldots \cdot (s_it)^2 \cdot (a_1t)^{m-2i}(a_2t)^{m-2i} = 0.$$

(9) *The point-quadric Q_i contains N_m and all its osculating spaces up to and including those of dimensions $i-1$; it also contains the $2(m-2i)$ osculating spaces of dimension i whose parameters t are determined by $(a_1t)^{m-2i}(a_2t)^{m-2i} = 0$.*

A dual statement holds for the S_{m-1}-quadric whose form is

$$Q_i = (\bar{x}_1\bar{x}_2)^{2i}(\alpha_1x_1)^{m-2i}(\alpha_2x_2)^{m-2i}.$$

To interpret the Gordan expansion of the symmetric form, we write in the usual way

$$(10) \quad (a_1x_1)^m(a_2x_2)^m = \sum_{i=0}^{i=\frac{m}{2};\,\frac{m}{2}-1} \frac{\binom{m}{2i}\binom{m}{2i}}{\binom{2m-2i+1}{2i}}(x_1x_2)^{2i}\{(a_1a_2)^{2i}(a_1x_1)^{m-2i}(a_2x_1)^{m-2i}\}_{\substack{m-2i\\x_2}}$$

where the subscript of the brace indicates the $(m-2i)$-th polar of the enclosed form as to x_2. Denoting the various terms of the expansion and the corresponding quadrics by Π_i, we have

$$(11) \qquad (a_1x_1)^m(a_2x_2)^m = \Pi_0 + \Pi_1 + \Pi_2 + \ldots + \Pi_{\frac{m}{2};\,\frac{m-1}{2}}.$$

Evidently the form of the last term varies according as m is even or odd.

The same form (10) with the dual meaning is expanded in the same way and we have

$$(12) \qquad (a_1\bar{x}_1)^m(a_2\bar{x}_2)^m = P_0 + P_1 + P_2 + \ldots + P_{\frac{m}{2};\,\frac{m-1}{2}}.$$

The quadric Π_i is of the type Q_i described in (9) but is further specialized by the fact that its form is a product of $(xy)^{2i}$ and a *polarized* form. This further condition is equivalent to the apolarity of the form with any form containing the factor $(xy)^{2i+2k}$, where $k \gtrless 1$. Thus a geometric statement of the expansion is:

(13) *With regard to a given norm-curve N_m, any quadric can be expressed uniquely as a sum of* $\frac{m}{2}+1$; $\frac{m-1}{2}+1$ *quadrics Π_i each of which is of the type described in* (9) *and furthermore is apolar to every quadric of the type P_{i+k} where $k>0$.*

The special case of this to be employed is

(14) *A symmetric form $(a_1x_1)^m(a_2x_2)^m$ is a polarized* $2m$-ic *when and only when the corresponding quadric is apolar to all the quadrics inscribed to the norm-curve.*

To express certain conditions it is necessary to have the reciprocal equation or form of the point quadric given by the form $(a_1x_1)^m(a_2x_2)^m$. As usual, we proceed as follows: The polar of a point $(c\bar{x})^m$ which lies on a space $(dx)^m$ as to the quadric is a space $(ex)^m$; that is,

$$\begin{aligned}(a_1c)^m(a_2x_2)^m &\equiv \rho(ex_2)^m,\\ (dc)^m &= 0.\end{aligned}$$

Eliminating the $m+2$ quantities $c_0, \ldots, c_m, \rho$ from the $m+2$ equations given by the identity and the equation, there results the reciprocal quadric polarized as to the spaces $(dx)^m$ and $(ex)^m$. The result expressed symbolically in terms of the coefficients of $(a_1x_1)^m(a_2x_2)^m=(a_1'x_1)^m(a_2'x_2)^m= \ldots$ is

(15) $$(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m \equiv (a_1\bar{x}_1)(a_1'\bar{x}_1)\ldots(a_1^{(m-2)}\bar{x}_1)(a_2\bar{x}_2)(a_2'\bar{x}_2)\ldots (a_2^{(m-2)}\bar{x}_2)\,\Pi(a_1^{(i)}a_1^{(k)})\,\Pi(a_2^{(i)}a_2^{(k)}),$$

where $i, k=0, 1, \ldots, m-2$ and $i \neq k$.

The discriminant of the given quadric can be found at once as the apolarity condition of $(a_1x_1)^m(a_2x_2)^m$ and $(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m$. To within a numerical factor its value is

(16) $$D=\Pi(a_1^{(i)}a_1^{(k)})\,\Pi(a_2^{(i)}a_2^{(k)}) \qquad i, k=0, 1, \ldots, (m-1) \text{ and } i\neq k.$$

Let the symmetric form be the polar of a $2m$-ic, $(hx)^{2m}$. Since $(hx)^{2m}$ can be expressed in ∞^1 ways as a sum of $(m+1)$ $2m$-th powers, say

$$(hx)^{2m}=\lambda_0(r_0x)^{2m}+\lambda_1(r_1x)^{2m}+\ldots+\lambda_m(r_mx)^{2m},$$

the symmetric form can also be expressed in ∞^1 ways as

$$(17) \qquad (hx_1)^m(hx_2)^m = \lambda_0(r_0x_1)^m(r_0x_2)^m + \lambda_1(r_1x_1)^m(r_1x_2)^m + \ldots.$$

Hence the quadric is in the form $\sum_{i=0}^{i=m} \lambda_i(R_iX_1)(R_iX_2)$, where the space R_i osculates N_m at the point whose parameter is r_i. Thus the spaces R_i form a self-polar $(m+1)$-edron of the quadric, and with reference to (14) we can state that

(18) *If a proper point quadric is apolar to all the quadrics inscribed to N_m, it has ∞^1 self-polar $(m+1)$-edra whose spaces osculate N_m.*

As shown above in detail for the forms $H_{0,n-k}$ and $H_{1,n-k}$, the general symmetric form $H_{k,n-k}$ can be viewed as a spread in S_{n-k} of order $k+1$ and dimensions $n-k-1$. The general spread can be represented in this way by a symmetric form. The spread is supposed taken in points or reciprocally according as the variables are not or are dashed.

We shall define an *antorthic* r-point of a point spread of order t* to be a set of r points such that any s points of the set are apolar to the spread of order t. such a set will be indicated by the symbol $A^r_{s,t}$. Of course $s \overline{\gtrless} r$ and $s \overline{\gtrless} t$. If the $H_{k,n-k}$ has an involutive set as defined earlier, this set can be interpreted as an $A^n_{k+1,k+1}$ of the spread determined by the form lying on the norm-curve N_{n-k}. One of the problems to be considered concerns the distribution of these $A^n_{k+1,k+1}$ upon norm-curves.

§ 3. *The Parametric† Spread of $H_{k,n-k}$.*

In $H_{k,n-k}$, let $x_1, x_2, \ldots, x_{k+1}$ be $k+1$ points on a norm-curve in a space S_{k+1}. The set of $k+1$ values x_i determine a point X in S_{k+1}. If the set be chosen so that $H_{k,n-k}=0$, the point X lies on a spread in S_{k+1} of order $n-k$ and dimensions k. Again the dual interpretation is indicated by dashed variables. The general spread in S_{k+1} can be represented in this way. For if in the equation of the spread the coordinates of the variable point X be expressed in terms of the symmetric combinations of the parameters $x_1, x_2, \ldots, x_{k+1}$, the equation takes the form $H_{k,n-k}=0$.

* An *orthic set* of r linear forms (German, "Pol-r-Seite") has been defined as a set in terms of whose powers a given form can be expressed. The antorthic r-point (German "Pol-r-Eck") may determine an orthic set by taking all the linear forms incident with the r points.

† The representation here given is the one usually employed: Meyer's *Apolarität* or Grace and Young.

An *inscribed n-space* of the spread is a set of n linear spaces S_k in S_{k+1} any $k+1$ of which meet in a point of the spread. If $H_{k,n-k}$ has an involutive set, the set represents an n-space inscribed in the spread and circumscribed to the norm-curve, N_{k+1}.

The spread in question will be denoted by Ψ_k^{n-k} of the form $H_{k,n-k}$; the spread of § 2, by Φ_{n-k-1}^{k+1} of $H_{k,n-k}$. Since each is general any spread can be represented in either way. Thus the Ψ_k^{n-k} of $H_{k,n-k}$ is the Φ_k^{n-k} of a symmetric form $H_{n-k-1,k+1}$. This form is obtained from $H_{k,n-k}$ by replacing the symmetric combinations of $x_1, x_2, \ldots, x_{k+1}$ by the coefficients of a binary form $(cx)^{k+1}=(c'x)^{k+1}=\ldots$; in the result of degree $n-k$ in the coefficients of $(cx)^{k+1}$, the coefficients of $n-k$ forms $(c_1x)^{k+1}, \ldots, (c_{n-k}x)^{k+1}$ are introduced by the Aronhold process; and finally the cofficients of the $n-k$ forms are replaced by $n-k$ sets of variables $x_1, x_2, \ldots, x_{n-k}$. By the inverse process the Φ_{n-k-1}^{k+1} of $H_{k,n-k}$ can be expressed as the Ψ_{n-k-1}^{k+1} of a symmetric form $H_{n-k-1,k+1}$. Only when $n=2k+1$ do the spreads Φ and Ψ of the *same* form $H_{k,n-k}$ have the same order and dimension. Though still distinct, they are covariants of each other and the norm-curve.

§ 4. *The Symmetric Form,* $H_{1,n-1}$.

The Φ_{n-2}^2 of $H_{1,n-1}=(a_1x_1)^{n-1}(a_2x_2)^{n-1}$ is a quadric in S_{n-1} considered with respect to a norm-curve N_{n-1}. If $H_{1,n-1}$ has an involutive set of n points, there is on N_{n-1} an antorthic set $A_{2,2}^n$ of Φ_{n-2}^2. Hence Φ in reciprocal form is apolar to all the point quadrics containing N_{n-1}. According to the dual of (18) there are then $\infty^1 A_{2,2}^n$ of Φ on N_{n-1}. Hence $H_{1,n-1}$ has ∞^1 involutive sets; *i. e.,*

(19) *If the symmetric form $H_{1,n-1}$ has a single involutive set of n points, it has ∞^1 sets and is an $I_{1,n-1}$.*

The Ψ_1^{n-1} of $H_{1,n-1}$ is a curve of order $n-1$ in S_2 with reference to a conic N_2. If $H_{1,n-1}$ has an involutive set, there is an n-line inscribed in Ψ and circumscribed about N_2. The translation of (19) is:

(20) *If an n-line circumscribed about a conic is inscribed in a curve of order $n-1$, there are ∞^1 n-lines circumscribed about the conic and inscribed in the curve whose n points of contact form an $I_{1,n-1}$ on the conic.*

The theorem of Stroh* that, after multiplication by an invariant, all the

* *Math. Annalen*, Vol. XXXIV, p. 323.

combinants of two binary forms of order n can be expressed as covariants of a single binary form of order $2(n-1)$ is also evident. For the condition that $H_{1,n-1}$ represent an involution is that the quadric Φ^2_{n-2} *in reciprocal form* be apolar to all the point-quadrics containing N_{n-1}. According to (14) this reciprocal form is then merely a polarized form of order $2(n-1)$, say $(\alpha x_1)^{n-1}(\alpha x_2)^{n-1}$. But from the reciprocal form, the original form $H_{1,n-1}$ can be recovered, multiplied however by the $(n-2)$th power of the discriminant of Φ which is the resultant of the two given binary forms. Thus the fundamental combinant, $H_{1,n-1}$, is itself a covariant of $(\alpha x)^{2(n-1)}$. The analytical results of Stroh follow readily from formulae (15) and (16).

§ 5. *Theorems Concerning Involutive Sets.*

(21) *The symmetric form $H_{k,n-k}$ is an $I_{k,n-k}$ if, for arbitrarily assigned values of $k-1$ variables $x_1, x_2, \ldots, x_{k-1}$, the form $H_{1,n-k}$ in the other two variables, x_k and x_{k-1}, is an $I_{1,n-k}$.*

Let the values $y_1, \ldots, y_k$ of k of the variables determine the values $z_{k+1}, z_{k+2}, \ldots, z_n$ of the remaining variable; *i. e.*, the $k+1$ values $y_1, \ldots, y_k$, z_{k+l}, where $l=1, 2, \ldots, n-k$, satisfy $H_{k,n-k}=0$. Since, for the given values $y_1, y_2, \ldots, y_{k-1}$, the form is an $I_{1,n-k}$, then also $y_1, y_2, \ldots, y_{k-1}, z_{k+l}, z_{k+m}$ satisfy $H_{k,n-k}=0$. Hence the values $y_1, \ldots, y_{k-1}, z_{k+l}$ determine the values y_k and z_{k+m} where $m \neq l$. Again, for given values $y_1, \ldots, y_{k-2}, z_{k+l}$, the form is an $I_{1,n-k}$; hence the $(k-2)$ y's and any three z's satisfy $H_{k,n-k}=0$. By a repetition of this process one can show that any $k+1$ values selected from the y's and z's satisfy $H_{k,n-k}=0$; whence the y's and z's constitute an involutive set and all sets determined by the form are involutive.

Writing, as before,

$$H_{k,n-k}=(a_1x_1)^{n-k}(a_2x_2)^{n-k}\ldots(a_{k+1}x_{k+1})^{n-k},$$

we have already determined the condition that the form

$$H_{1,n-k}\equiv(a_kx_k)^{n-k}(a_{k+1}x_{k+1})^{n-k}$$

be an $I_{1,n-k}$; namely, if $(\alpha_k\bar{x}_k)^{n-k}(\alpha_{k+1}\bar{x}_{k+1})^{n-k}$ of degree $n-k$ in the coefficients of $H_{1,n-k}$ be the quadric reciprocal to $H_{1,n-k}$ (in the sense of § 2), then the identical vanishing of $(\alpha_k\alpha_{k+1})^2(\alpha_k\bar{x}_k)^{n-k-2}(\alpha_{k+1}\bar{x}_{k+1})^{n-k-2}$ is the required con-

dition. Expressing that this condition is satisfied for all values of x_1, x_2, ..., x_{k-1}, we have

$$(22)\quad (a_1x_1)^{n-k}(a_1'x_1)^{n-k}\ldots(a_1^{(n-k-1)}x_1)^{n-k}\ldots(a_{k-1}x_{k-1})^{n-k}(a_{k-1}'x_{k-1})^{n-k}$$
$$\ldots(a_{k-1}^{(n-k-1)}x_{k-1})^{n-k}\,.\,(a_k a_{k+1})^2(a_k\bar{x}_k)^{n-k-2}(a_{k+1}\bar{x}_{k+1})^{n-k-2}\equiv 0.$$

Hence

(23) *The identical vanishing of the form* (22) *of degree $n-k$ in the coefficients of $H_{k,n-k}$, symmetrical of order $(n-k)^2$ in $k-1$ variables, and symmetrical of order $n-k-2$ in two variables, is the necessary and sufficient condition that $H_{k,n-k}$ be an $I_{k,n-k}$.*

Naturally this condition does not apply to the case $n-k=1$. For then mere symmetry is sufficient to require an $I_{k,1}$ and the form $H_{k,1}$ is a completely polarized $(k+1)$-ic. The most convenient application is to the case $n-k=2$. Then the two variables $\bar{x}_k$ and $\bar{x}_{k+1}$ disappear and the condition is a symmetric form, $H_{k-2,4}=0$. Most of what follows is concerned with this case.

Specializing the results just obtained we have

(24) *If $H_{1,2}=(a_1x_1)^2(a_2x_2)^2=(a_1'x_1)^2(a_2'x_2)^2$ has one involutive set, it has ∞^1 sets and is an $I_{1,2}$. The invariant condition for this is*

$$(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2')+(a_1a_2')(a_1'a_2)\}=0.$$

From this theorem we have further:

(25) *The form $H_{2,2}\equiv(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2=(a_1'x_1)^2(a_2'x_2)^2(a_3'x_3)^2=\ldots.$ has in general a single involutive set given by the quartic*

$$(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2')+(a_1a_2')(a_1'a_2)\}(a_3x)^2(a_3'x)^2=0.$$

If the form has more than one involutive set, it has ∞^2 sets and is an $I_{2,2}$. The invariant condition for this is the identical vanishing of the quartic.

For if y_1, y_2, y_3, y_4 are the roots of the quartic, they are the values for which $H_{2,2}$ becomes an $I_{1,2}$ (24). Let $H_{2,2}$, when $x_1=y_1$ and $x_2=y_2$, determine the values z_3 and z_4 of x_3. Since, when $x_1=y_1$, the form is an $I_{1,2}$ and is satisfied by y_1, y_2, z_3 and y_1, y_2, z_4, it is also satisfied by y_1, z_3, z_4. Similarly it is satisfied by y_2, z_3, z_4. Hence y_1, y_2, z_3, z_4 are an involutive set of $H_{2,2}$ and for any one of the four, $H_{2,2}$ reduces to an $I_{1,2}$. That is, z_3 and z_4 are y_3 and y_4. If $H_{2,2}$ has more than one involutive set, the quartic must vanish identically. But according to (23), $H_{2,2}$ is then an $I_{2,2}$.

Again, let

$$H_{3,2}\equiv(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2(a_4x_4)^2=(a_1'x_1)^2(a_2'x_2)^2(a_3'x_3)^2(a_4'x_4)^2=\ldots..$$

If, for assigned values of two of the variables x_1 and x_2, $H_{3,2}$ reduces to an $I_{1,2}$, x_1 and x_2 satisfy the symmetric relation

$$\overline{H}_{1,4} = (a_3a_3')(a_4a_4')\{(a_3a_4)(a_3'a_4') + (a_3a_4')(a_3'a_4)\}(a_1x_1)^2(a_1'x_1)^2(a_2x_2)^2(a_2'x_2)^2 = 0.$$

We note first that every involutive set of $H_{3,2}$ is an involutive set of $\overline{H}_{1,4}$. Conversely, every involutive set of $\overline{H}_{1,4}$ is an involutive set of $H_{3,2}$. For if $x_1, \ldots, x_5$ is an involutive set of $\overline{H}_{1,4}$, and if the value x_1 be assigned in $H_{3,2}$, reducing it to an $H_{2,2}$, the single involutive set of this $H_{2,2}$ is $x_2, \ldots, x_5$ (25). Hence x_1 and any three of $x_2, \ldots, x_5$ satisfy $H_{3,2} = 0$. Similarly with regard to x_2, etc., and the set is involutive. If $\overline{H}_{1,4}$ has one involutive set, it has ∞^1, and we conclude:

(26) *$H_{3,2}$ has in general no involutive set. If it has one involutive set, it has ∞^1 which form an $I_{1,4}$. The invariant condition for this is the condition that $\overline{H}_{1,4}$ be an $I_{1,4}$ [see (22) and (23)]. If $H_{3,2}$ has an involutive set not contained in the $I_{1,4}$, it has ∞^3 sets and is an $I_{3,2}$, the invariant condition for which is $\overline{H}_{1,4} \equiv 0$.*

From induction there follows:

(27) *The form $H_{k,2}$ has the same involutive sets as the form $\overline{H}_{k-2,4}$ whose identical vanishing is the condition that $H_{k,2}$ be an $I_{k,2}$.*

Thus a study of the forms $H_{k,2}$ when $k > 2$ involves that of the symmetric forms of order 4 in more than two variables.

§ 6. *The Form $H_{1,2}$ as a Ternary Quadratic.*

The binary quadratic $(a\bar{x})^2 = a_2^2x_1^2 - 2a_2a_1\bar{x}_1\bar{x}_2 + a_1^2\bar{x}_2^2$ represents a point X, and the quadratic $(ax)^2$ a line U, in a plane when we put

$$(28) \qquad \begin{array}{lll} X_0 = a_2^2, & X_1 = -2a_1a_2, & X_2 = a_1^2 \\ U_0 = a_1^2, & U_1 = a_1a_2, & U_2 = a_2^2. \end{array}$$

The condition that three points lie on a line or that three lines pass through a point is that the corresponding quadratics be in involution; the incidence condition is the apolarity condition. Employing the usual symbolic notations for the binary and ternary domains, the formulae are

$$(29) \qquad \begin{aligned} (XYZ) &= 2(ab)(ac)(bc) \\ (UVW) &= (ab)(ac)(bc) \\ U_X &= (ab)^2. \end{aligned}$$

Ternary forms of degree higher than the first are always polarized; thus the norm-conic is

$$(30)\quad \begin{aligned} N_X N_Y &= (2X_2X_0 - \tfrac{1}{2}X_1^2)_Y = X_2Y_0 + X_0Y_2 - \tfrac{1}{2}X_1Y_1 = (x_1x_2)^2 \\ U_N V_N &= (2U_2U_0 - 2U_1^2)_V = U_2V_0 + U_0V_2 - 2U_1V_1 = (x_1x_2)^2. \end{aligned}$$

Small letters can be used for both domains since the difference in symbols distinguishes them.

The general conic, its line form, and its discriminant are at once written from (29):

$$(31)\quad \begin{aligned} a_x a_y &= (a_1x_1)^2(a_2x_2)^2 = (a_1x_2)^2(a_2x_1)^2 = (a_1'x_1)^2(a_2'x_2)^2 = \ldots, \\ (aa'u)(aa'v) &= (a_1a_1')(a_2a_2')(a_1\bar{x}_1)(a_1'\bar{x}_1)(a_2\bar{x}_2)(a_2'\bar{x}_2), \\ (aa'a'')^2 &= (a_1a_1')(a_1a_1'')(a_1'a_1'')(a_2a_2')(a_2a_2'')(a_2'a_2''). \end{aligned}$$

Thus $H_{1,2}$ appears as a conic whose apolarity relations with the norm-conic N are

$$(32)\quad \begin{aligned} a_N^2 &= (a_1a_2)^2 \\ (aa'N)^2 &= \tfrac{1}{2}(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2') + (a_1a_2')(a_1'a_2)\}. \end{aligned}$$

If $H_{1,2}$ has one involutive sets, it has ∞^1 sets. Then $(aa'N)^2 = 0$ and $H_{1,2}$ is an $I_{1,2}$. The triads of $I_{1,2}$ are antorthic sets $A_{2,2}^3$ (*i. e.*, self-polar triangles) of the conic $H_{1,2}$ which lie on N.

If, for the $I_{1,2}$, $a_N^2 = 0$, the conics N and $H_{1,2}$ meet in a set of four points equianharmonic on either conic. The $I_{1,2}$ is then the second polar system of this quartic.

If, for the $I_{1,2}$, $(aa'a'')^2 = 0$, $H_{1,2}$ has a double point on N with parameter t, say. Then the form $H_{1,2}$ is $(x_1 - t)(x_2 - t) \,.\, (\alpha_1x_1)(\alpha_2x_2)$, where $(\alpha_1x_1)(\alpha_2x_2)$ is a form $H_{1,1}$ such that $(\alpha_1x)(\alpha_2x) = 0$ fixes the two other meets of $H_{1,2}$ and N. In this case the $I_{1,2}$ degenerates into the neutral point t and an $I_{1,1}$.

If, for the $I_{1,2}$, $(aa'u)(aa'v)$ vanishes identically, the conic $H_{1,2}$ is the square of a line which meets N in points whose parameters are s and t. The form $H_{1,2}$ is then $(x_1 - s)(x_1 - t)(x_2 - s)(x_2 - t)$, and the $I_{1,2}$ degenerates into an arbitrary point and the neutral points s and t.

§ 7. *The Form $H_{2,2}$ as a Ternary Cubic.*

The spread of $H_{2,2}$ as defined in § 2 is a plane cubic, $a_x^3 = a_x'^3 = a_x''^3 = \ldots$, and we write

$$(33)\quad f = a_x a_y a_z = (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2.$$

Involutive sets of $H_{2,2}$ are sets of 4 points on the norm-conic N such that any 3 of the 4 are apolar to the cubic, *i. e.*, antorthic sets $A^4_{3,3}$.* Using the first property of (25), we find that

(34) *On any conic there is, in general, a single antorthic 4-point of a given cubic.*

The spread of $H_{2,2}$ as defined in § 3 is a space quadric, and the translated theorem is:

(35) *There is, in general, a single tetrahedron circumscribed about a cubic space curve and inscribed in a quadric.*†

If the point x (or x_1 on N) belongs to the antorthic set, the polar conic of x as to f, when taken in lines, is apolar to N, or, symbolically, $(aa'N)^2a_xa'_x = 0$. The meets of this conic and N^2_x form the unique antorthic set.

If the $A^4_{3,3}$ is not unique, the $H_{2,2}$ is an $I_{2,2}$, and N and f are subject to the relation

$$(aa'N)^2a_xa'_x = \lambda N^2_x, \tag{36}$$

which expresses, in the ternary domain, that the polar conic as to f of any point x on N, when taken in line form, is apolar to N in point form; and, in the binary domain, that for any value of x, the form $H_{2,2}$ reduces to an $I_{1,2}$ and is therefore an $I_{2,2}$.

The conics N^2_x which satisfy the relation (36) have been investigated in companion papers by Professors White and Gordan.‡ The former finds, corresponding to the values $\lambda = \pm\sqrt{\tfrac{1}{6}S}$, two nets of conics, the polar conics of two irrational covariants of f,

$$A \equiv A^3_x = \Delta + \sqrt{\tfrac{1}{6}S}f, \text{ and } B = B^3_x = \Delta - \sqrt{\tfrac{1}{6}S}f. \tag{37}$$

These conics on each of which lie ∞^2 antorthic sets of f which constitute an $I_{2,2}$ will be called *involution conics*.

An $A^4_{3,3}$ is a set of four points subject to four relations; *i. e.*, there are ∞^4 $A^4_{3,3}$. In general, an $A^4_{3,3}$ is determined uniquely when two of its points, say x and y, are given. For the other two points, z and t must lie on the polar line of

* These sets have been discussed by Caporali, *Memorie*, p. 51, § 5.

† Theorem of Meyer, *Apolarität*, p. 279 (y_1). The parametric representation is treated there quite fully, and further results in this direction will not be given.

‡ *Trans. American Math. Soc.*, Vol. I, pp. 1 and 9. Other references are given there. The notation for the comitants of f is that of Gordan.

xy as to f and must be harmonic with the pairs of points in which this polar line meets the polar conics of x and y. Through x and y there passes a definite polar conic of A and of B. Since each is an antorthic conic, the polar line of xy as to f must meet each in a pair of points which form with x and y an $A^4_{3,3}$ of f. Hence these conics meet in the $A^4_{3,3}$, x, y, z, t.

(38) *The ∞^4 $A^4_{3,3}$'s of a cubic f are distributed ∞^2 at a time on the ∞^2 polar conics of A; and in the same way on the ∞^2 polar conics of B. On a fixed polar conic of the one cubic, the $A^4_{3,3}$'s lie in an $I_{2,2}$ which is cut out by all the polar conics of the other cubic.*

Assuming henceforth that $H_{2,2}$ is an $I_{2,2}$ on an involution conic N_x^2 we have the following parallel between the comitants (polarized) of f and the comitants of the form $I_{2,2}$, the transition being made at once by the use of (29):

$$
\begin{aligned}
(39)\quad N &= N_x N_y &&= (x_1x_2)^2,\\
f &= a_x a_y a_z &&= (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2,\\
\theta &= (aa'u)(aa'v)a_x a'_y &&= (a_1x_1)^2(a'_1x_2)^2(a_2a'_2)(a_3a'_3)(a_2\bar{x}_1)(a'_2\bar{x}_1)(a_3\bar{x}_2)(a'_3\bar{x}_2),\\
\Delta &= (aa'a'')^2 a_x a'_y a''_z &&= (a_1x_1)^2(a'_1x_2)^2(a''_1x_3)^2(a_2a'_2)(a_2a''_2)\\
&&&\qquad (a'_2a''_2)(a_3a'_3)(a_3a''_3)(a'_3a''_3)\\
&&&= (\delta_1x_1)^2(\delta_2x_2)^2(\delta_3x_3)^2.
\end{aligned}
$$

Then the relation (36) becomes, in the binary domain,

$$(40)\quad \tfrac{1}{2}(a_1x_1)^2(a'_1x_2)^2(a_2a'_2)(a_3a'_3)[(a_2a_3)(a'_2a'_3)+(a_2a'_3)(a'_2a_3)] = \sqrt{\tfrac{1}{6}S}\,(x_1x_2)^2.$$

Expanding the left-hand member in powers of (x_1x_2), the first term is a polar of the binary quartic in (25) which is now identically zero. The second term is an invariant multiplied by $(x_1x_2)^2$, and equating coefficients we have

$$(41)\quad +\sqrt{\tfrac{1}{6}S} = \tfrac{1}{6}(a_1a')^2(a_2a'_2)(a_3a'_3)[(a_2a_3)(a'_2a'_3)+(a_2a'_3)(a'_2a_3)].$$

Thus *the irrational invariant $\sqrt{\tfrac{1}{6}S}$ of f is a rational invariant of the form $I_{2,2}$.* This is to be expected, since the irrationality is really adjoined by the choice of N_x^2 as an involution conic. Noting that a polar conic of $A = \Delta + \sqrt{\tfrac{1}{6}S}f$ is one which satisfies the relation (36) for the value $\lambda = +\sqrt{\tfrac{1}{6}S}$, since

$$A_x(aa'A)^2a_ya'_y = +\sqrt{\tfrac{1}{6}S}\,[\Delta_x\Delta_y^2 + \sqrt{\tfrac{1}{6}S}f_xf_y^2] = +\sqrt{\tfrac{1}{6}S}\,A_xA_y^2,^*$$

we can write

$$
\begin{aligned}
(42)\quad A &= \Delta + \sqrt{\tfrac{1}{6}S}f = (\delta_1x_1)^2(\delta_2x)^2(\delta_3x_3)^2 + \sqrt{\tfrac{1}{6}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2,\\
B &= \Delta - \sqrt{\tfrac{1}{6}S}f = (\delta_1x_1)^2(\delta_2x)^2(\delta_3x_3)^2 - \sqrt{\tfrac{1}{6}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2\\
&= (b_1x_1)^2(b_2x_2)^2(b_3x_3)^2,
\end{aligned}
$$

* The symbolic calculations not explicitly given follow readily from the formulae of Gordan (*loc. cit.*) or from those in Clebsch-Lindemann, Vol. II (French edition).

where of course $\sqrt{\frac{1}{6}S}$, when it occurs in a binary expression, is understood to be the binary invariant of (41). *The choice of the positive sign in* (41) *requires that* A *in* (42) *be the cubic whose polar conic is* $N \equiv (x_1x_2)^2$.

When the $I_{2,2}$ on N is given, the cubic f, the value $+\sqrt{\frac{1}{6}S}$, and the point P whose polar as to A is N are uniquely determined. Conversely, when f, the value $+\sqrt{\frac{1}{6}S}$, and the point P are given, the antorthic conic N and the $I_{2,2}$ on it are uniquely determined. Since the system of invariants of f and P is the system of covariants of f, we conclude that

(43) *The rational invariant theory of the involution form* $I_{2,2}$ *coincides with the rational covariant theory of the ternary cubic* f *after the adjunction of* $\sqrt{\frac{1}{6}S}$.

Thus the invariants of $I_{2,2}$ are of two types according as they correspond to invariants or covariants of f.

The best illustration of the $I_{2,2}$ is the system of intersections of a rational plane quartic curve by lines of the plane, and we shall look particularly for the invariants which correspond to obvious peculiarities of the quartic curve. Since the quartic has three double points, the $I_{2,2}$ has three neutral pairs, and on N there are three pairs of points whose polar line is indeterminate; *i. e.*, three pairs of corresponding points on Δ.

(44) *The two nets of antorthic conics can be defined as the conics which cut the Hessian of* f *in three pairs of corresponding points.*

A direct proof of this is immediate. If $A_PA_x^2$ is any polar conic of A, it satisfies the relation $A_P(aa'A)^2a_xa'_x = \sqrt{\frac{1}{6}S}\,A_PA_x^2$. But if x is on Δ, its corresponding point y is given by $(aa'u)^2a_xa'_x = \rho u_y^2$. Thus if x on Δ satisfies $A_PA_x^2 = 0$, y does the same.

That these nets contain all such conics is seen from the elliptic parametric representation of Δ. Let corresponding points u and u' satisfy the relation $u - u' \equiv \frac{\omega}{2}$ where ω and $\bar{\omega}$ are the independent periods. Then u, v, w, and their corresponding points, lie on a conic if either

$$\frac{\omega}{4} + u + v + w \equiv \begin{cases} 0 \\ \omega/2 \end{cases}$$

or

$$\frac{\omega}{4} + u + v + w \equiv \begin{cases} \bar{\omega}/2 \\ (\bar{\omega} + \omega)/2. \end{cases}$$

In the first case u and v determine a unique pair w, w' and the conics lie in a net. In the second case a similar net is obtained which is distinct from the first. These two nets must be $A_P A_x^2$ and $B_P B_x^2$.

Through u and u' there passes a pencil of each net; hence the line $\overline{uu'}$ is part of a degenerate conic in each net as well as in the net of polar conics of f. We obtain then the theorem of Professor White:

(45) *The relation of the cubics f, A, and B is mutual. They are three cubics with a common Cayleyan.*

Moreover, referring to (44):

(46) *The polar conics of A, in particular N, cut Δ and*

$$\Delta_B = -\tfrac{1}{3}\left(T - \sqrt{\tfrac{1}{6}S^3}\right)\left(\Delta + 3\sqrt{\tfrac{1}{6}S}f\right),$$

the Hessian of B, each in three pairs of corresponding points.

Continuing further the parallel between the ternary and binary forms, we have

$$
\begin{aligned}
(47) \qquad s = u_s^3 &= (aa'a'')(aa'u)(aa''u)(a'a''u) \\
&= (a_1a_1')(a_1a_1'')(a_1'a_1'')(a_2a_2')(a_3a_3')(a_2''a_3'')(a_2\bar{x}_1)(a_2'\bar{x}_1)(a_3\bar{x}_2) \\
&\qquad (a_3'\bar{x}_2)(a_2''\bar{x}_3)(a_3''\bar{x}_3) \\
&= (\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2, \\
a_s^3 = S &= (\sigma_1 a_1)^2(\sigma_2 a_2)^2(\sigma_3 a_3)^2, \\
\Delta_s^3 = T &= (\sigma_1\delta_1)^2(\sigma_2\delta_2)^2(\sigma_3\delta_3)^2, \\
u_P = N_s^2 u_s &= (\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2.
\end{aligned}
$$

By comparing the value of S just given with the value of $\sqrt{\tfrac{1}{6}S}$ in (41), a symbolic identity is obtained.

From the ternary identity

$$A_P A_s^2 u_s = \tfrac{1}{3}\left(T + \sqrt{\tfrac{1}{6}S^3}\right)u_P$$

we see that *the point P in (47) is the pole of the norm-conic as to A.* Forming in the binary notation the polar conic of u_P as to A in (42), we find, since the conic is N_x^2,

$$(\sigma_1\sigma_2)^2(\sigma_3\delta_3)^2(\delta_1x_1)^2(\delta_2x_2)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3a_3)^2(a_1x_1)^2(a_2x_2)^2 = \tfrac{1}{3}\left(T + \sqrt{\tfrac{1}{6}S^3}\right)(x_1x_2)^2.$$

Expanding the left-hand member into Gordan's series and equating coefficients, we obtain two identities,

$$(48) \qquad (\sigma_1\sigma_2)^2(\sigma_3\delta_3)^2(\delta_1 x)^2(\delta_2 x)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3 a_3)^2(a_1 x)^2(a_2 x)^2 = 0,$$

$$(49) \qquad (\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3) + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1 a_2)^2(\sigma_3 a_3)^2 = T + \sqrt{\tfrac{1}{6}S^3}.$$

Let $T + S\sqrt{\tfrac{1}{6}S} = R_1$ and $T - S\sqrt{\tfrac{1}{6}S} = R_2$; *i. e.*, the discriminant of f is $R = R_1 R_2 = T^2 - \tfrac{1}{6}S^3$.

If in (49) the values of T, S, and $\sqrt{\tfrac{1}{6}S}$ be expressed by means of (47) and (41), we obtain *a syzygy among the invariants of the sixth degree of the form* $I_{2,2}$; namely,

$$(50) \qquad \begin{aligned}(\sigma_1\delta_1)^2(\sigma_2\delta_2)^2(\sigma_3\delta_3)^2 &+ \sqrt{\tfrac{1}{6}S}(\sigma_1 a_1)^2(\sigma_2 a_2)^2(\sigma_3 a_3)^2 \\ = (\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)^2 &+ \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^1(a_1 a_2)^2(\sigma_3 a_3)^2.\end{aligned}$$

That (50) is not a mere identity is verified by expressing the last two terms as invariants of the cubic f and point P while the first two terms are invariants of f alone. Consider the form

$$u_s^2 a_s a_x^2 = (\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3 a_3)^2(a_1 x_1)^2(a_2 x_2)^2.$$

The result of operating on the right side with $(x_1 x_2)^2$ [on $\bar{x}_1$ and $\bar{x}_2$] and with $(\bar{x}_1\bar{x}_2)^2$ [on x_1 and x_2] is $\tfrac{1}{9}(\sigma_1\sigma_2)^2(\sigma_3 a_3)^2(a_1 a_2)^2$. Similarly operating on the left with the norm-conic in points and with the norm-conic in lines, *i. e.*, with

$$\frac{1}{R_1} A_P A_x^2 \quad \text{and} \quad \frac{1}{R_1^2} A_P A_P'(AA'u)^2,$$

we find

$$\frac{1}{R_1^3} a_s A_P A_s^2 A_P' A_P''(A'A_u'')^2 = \frac{1}{9}\frac{1}{R_1} a_P^3;$$

hence

$$(51) \qquad a_P^3 = R_1 \cdot (\sigma_1\sigma_2)^2(a_1 a_2)^2(\sigma_3 a_3)^2;$$

and similarly

$$2\sqrt{\tfrac{1}{6}S}a_P^3 - \Delta_P^3 = R_1 \cdot (\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)^2.$$

(52) *When the cubic f is non-singular, the invariant condition on the form $I_{2,2}$ that the pole P of the norm-conic N as to A shall lie on the curve $k_1 B + k_2\sqrt{\tfrac{1}{6}S}f$ is*

$$k_1(\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3) - (k_2 + k_1)\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1 a_2)^2(\sigma_3 a_3)^2 = 0,$$

or

$$k_2(\sigma_1\sigma_2)^2(b_1 b_2)^2(\sigma_3 b_3)^2 - k_2\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1 a_2)^2(\sigma_3 a_3)^2 = 0.$$

Another statement of the same condition is

(53) *The pole P of N as to A lies on the curve $k_1B+k_2\sqrt{\tfrac{1}{6}S}f$ if the polar line of N (in lines) as to the curve $k_1B-k_2\sqrt{\tfrac{1}{6}S}f$ passes through P.*

In formulae (41), (47) and (52) we have the expressions for the invariants of f and the covariants of its syzygetic pencil in terms of the invariants of the form $I_{2,2}$. Other covariants would probably be not less useful. The combinants of the syzygetic pencil ought to be particularly interesting with reference to a pair of quartic envelopes defined in the next paragraph.

The following ternary identities also will be used:

$$
\begin{aligned}
(54)\quad \Delta_A &= -\tfrac{1}{3}R_1(\Delta-3\sqrt{\tfrac{1}{6}S}f); & S_A &= \tfrac{2}{3}R_1^2; & \sqrt{\tfrac{1}{6}S_A} &= \pm\tfrac{1}{3}R_1.\\
\Delta_B &= -\tfrac{1}{3}R_2(\Delta+3\sqrt{\tfrac{1}{6}S}f); & S_B &= \tfrac{2}{3}R_2^2; & \sqrt{\tfrac{1}{6}S_B} &= \pm\tfrac{1}{3}R_2.\\
s_A &= \tfrac{2}{3}R_1\,.\,s; & A_A &= \tfrac{4}{3}R_1\,.\,\sqrt{\tfrac{1}{6}S}\,.\,f \text{ or } -\tfrac{2}{3}R_1\,.\,B.\\
s_B &= \tfrac{2}{3}R_2\,.\,s; & B_B &= -\tfrac{4}{3}R_2\,.\,\sqrt{\tfrac{1}{6}S}f \text{ or } -\tfrac{2}{3}R_2\,.\,A.
\end{aligned}
$$

These can be employed to obtain a convenient equation for the involution conics through given points x and y. The polar conic of A through x and y is

$$
\begin{aligned}
3(AA'A'')A_x^2A_y'^2A_z''^2 &= 2(xyz)\,.\,\Delta_{Ax}\Delta_{Ay}\Delta_{Az}+(s_Axy)(s_Axz)(s_Ayz)^*\\
&= -\tfrac{2}{3}R_1[(xyz)\{\Delta_x\Delta_y\Delta_z-3\sqrt{\tfrac{1}{6}S}a_xa_ya_z\}-(sxy)(sxz)(syz)].
\end{aligned}
$$

Since

$$(sxy)(sxz)(syz)=(aa'a'')a_x^2a_y'^2a_z''^2+4(aa'a'')a_xa_ya_x'a_z'a_y''a_z''$$

and

$$3(aa'a'')a_x^2a_y'^2a_z''^2=2(xyz)\Delta_x\Delta_y\Delta_z+(sxy)(sxz)(syz),$$

we find

$$(AA'A'')A_x^2A_y'^2A_z''^2=\tfrac{2}{3}R_1[2(aa'a'')a_xa_ya_x'a_z'a_y''a_z''+\sqrt{\tfrac{1}{6}S}a_xa_ya_z\,.\,(xyz)].$$

(55) *The involution conics through x and y are*

$$2(aa'a'')a_xa_ya_z'a_z'a_y''a_z''\pm\sqrt{\tfrac{1}{6}S}\,.\,(xyz)\,.\,a_xa_ya_z=0.$$

Since N is a polar conic of A, the net of polar conics of A cut out on N an $I_{1,3}$, and the lines joining the four points of a set of $I_{1,3}$ are lines of the Cayleyan. If a line of s cuts N in the points t and τ, the parametric equation of s (see § 3) is given by the *involution form* †

$$(56)\qquad (\sigma_1t)(\sigma_2t)(\sigma_3t)(\sigma_1\tau)(\sigma_2\tau)(\sigma_3\tau)=0.$$

---

* See Gordan, *loc. cit.*, p. 10. The minus sign in the formula given there should be changed.

† That the joins of the four points of a set of $I_{1,3}$ are lines of the Cayleyan shows that the Cayleyan is the Hessian of a cubic B to which N is apolar. From this there follows the apolarity of the nets of polar conics of A and B; see Gordan, § 10.

Finally, we may emphasize in the binary domain the translation of the mutual relation of f and B with respect to A and to their Cayleyan:

(57) *The involution form $I_{2,2}=f$ has a covariant involution form $\overline{I}_{2,2}=B$ o the third degree in the coefficients. The relation of the two is mutual, and the $\overline{I}_{2,2}$ formed for $\overline{I}_{2,2}$ is $I_{2,2}$ multiplied by the invariant $-\frac{4}{3}\sqrt{\frac{1}{3}S}R_2$. $I_{2,2}$ has also a covariant involution $I_{1,3}$ of the third degree. The $I_{1,3}$ formed for $\overline{I}_{2,2}$ is $I_{1,3}$ itself multiplied by the invariant $\frac{4}{3}R_2$.*

§ 8. *The $I_{2,2}$ and the Ternary Rational Quartic.*

The given $I_{2,2}$ is cut out on N by the polar conics of B, whence the polar lines of points on N as to B envelop a rational quartic R_4 whose lines correspond to the points of N and whose $I_{2,2}$ (the four lines of R_4 through a point) corresponds to the given $I_{2,2}$ on N. In the same way, the covariant $\overline{I}_{2,2}$ on N is cut out by the polar conics of f, and the polar lines of points on N as to f envelop a rational quartic $\overline{R}_4$ whose $I_{2,2}$ is $\overline{I}_{2,2}$. In this paragraph some phases of the mutual relation between R_4 and $\overline{R}_4$ will be discussed.

Since $B=(b_1x_1)^2(b_2x_2)^2(b_3x_3)^2$, the polar conic of the point $(q\bar{x})^2$ as to B is $(b_1q)^2(b_2x_2)^2(b_3x_3)^2$, which meets N in four points whose parameters are $(b_1q)^2(b_2x)^2(b_3x)^2=0$.

(58) *The given $I_{2,2}$ is the ∞^2 quartics, for variable q, $(b_1q)^2(b_2x)^2(b_3x)^2=0$. The covariant $\overline{I}_{2,2}$ is the ∞^2 quartics $(a_1q)^2(a_1x)^2(a_2x)^2=0$.*

When $(q\bar{x})^2$ is a point on N, say $(y\bar{x})^2$, the quartic $(a_1y)^2(a_2x)^2(a_3x)^2=0$ is found in $\overline{I}_{2,2}$. But in the given $I_{2,2}$ this is four values x, each of which, taken twice with y, belongs to $I_{2,2}$. Hence

(59) *The tangent at a point of R_4 cuts R_4 in four other points whose parameters form a set of $\overline{I}_{2,2}$, and the tangent at a point of $\overline{R}_4$ cuts $\overline{R}_4$ in four other points whose parameters form a set of $I_{2,2}$. More generally, there exist on R_4 sets of four points such that the parameters of the remaining two tangents of R_4 from each point lie in a pencil of quadratics; these sets of four points lie in the $\overline{I}_{2,2}$ of $\overline{R}_4$.*

Some special cases of the antorthic $A^4_{3,3}$ call for special treatment, in particular those which are not determined when two of their points, x and y, are given. In general, the two other points are obtained as the meets of the conic $(aa'a'')a_xa_ya'_xa'_za''_ya''_z=0$ and the line $a_xa_ya_z=0$. Since the conic is not evanescent, the construction fails if (1) the conic has the line for a factor or if (2) the

line is evanescent. Denote by $Q_{x,y}$ the quadratic involutory Cremona transformation whose corresponding points are apolar to the polar conics of x and y as to f.

In case (1), the polar lines of z, any point on the polar line of xy, as to the polar conics of x and y, must meet in a point t on this polar line. Hence this polar line corresponds to itself in $Q_{x,y}$ and must be a diagonal line of the two polar conics, *i. e.*, a line of the Cayleyan. The points z and t are harmonic with the vertices on the diagonal line which are poles of the line $\overline{xy}$. From (38), if through x and y there passes a definite polar conic of A, then through x and y there must pass a pencil of polar conics of B which meet A in pairs of point z, t. Hence the polar conic of A must be the lines $(xyz) = 0$ and $a_x a_y a_z = 0$. Or if u_A and v_A are a degenerate polar conic of A, each line has two of its poles as to f on the other line; the poles determine on each line a pencil of quadratics; and any member of the one pencil with any member of the other is an $A^4_{3,3}$. A similar statement holds for the degenerate polar conics of B. There are ∞^3 such $A^4_{3,3}$, ∞^1 on each proper involution conic. On N they lie in such a way that any point is contained in three $A^4_{3,3}$'s. If, in the $I_{1,3}$ of (56), t is given, three values of τ (τ_1, τ_2 and τ_3) are determined. The lines $t\tau_1$ and $\tau_2\tau_3$ form a polar conic of A, while $t\tau_1$ and the line paired with it as a polar conic of B fix an $A^4_{3,3}$ which contains t.

In case (2), the pair x, y whose $A^4_{3,3}$ is indeterminate is a pair of corresponding points on Δ. Through x and y there passes a pencil of polar conics of A and a pencil of polar conics of B. If N be taken as a fixed polar conic of A, the pencil of polar conics of B cuts out on N a quadratic involution, $I_{1,1}$.

If x and y lie on u and $Q_{x,y}$ is the Cremona involution defined earlier, then every pair of corresponding points on Δ is a pair of corresponding points in $Q_{x,y}$. Δ itself passes through the singular points of $Q_{x,y}$ which correspond on Δ to the three meets of Δ with u. The fixed points of $Q_{x,y}$ are the four poles of u. On any line v there is a pair of corresponding points in $Q_{x,y}$, say z, t, a pair whose polar line is u. On a conic there are in general four pairs of corresponding points in $Q_{x,y}$.

On the conic N let the three corresponding pairs of Δ be h_i, h'_i; let them be joined by lines H_i respectively; and let the vertex of the triangle opposite H_i be J_i; $i = 1, 2, 3$. Similarly let the three corresponding pairs, k_i, k'_i, of Δ_B on N be joined by lines K_i which meet in points L_i.

If u meets N in points x, y, the four pairs of corresponding points on N in

$Q_{x,y}$ are the three pairs h_i, h'_i and the pair z, t cut out by the polar line of xy. Thus the relation of the lines $\overline{xy}$ and $\overline{zt}$ is reciprocal; they are corresponding lines in an involutory Cremona quadratic line-transformation, $\bar{Q}$, whose singular lines are H_1, H_2, H_3. Since x, y, z, t together form an $A^4_{3,3}$, we see, on allowing x and y to coincide, that

(60) *$\bar{Q}$ transforms the tangents of N into the tangents of $\overline{R}_4$, the tangent at x on N being transformed into the polar line of x as to f.*

Since lines through J_i are transformed by $\bar{Q}$ into H_i, we find that

(61) *The neutral set, h_i, h'_i, forms with the point-pairs cut out on N by the pencil of lines through J_i sets of the $I^4_{2,2}$ on N.*

The Q_{h_i, h'_i} has for corresponding pairs any pair of points whose polar line is $\overline{h_i h'_i}$. Since there is on N ∞^1 such pairs, N is unaltered by Q_{h_i, h'_i}. Evidently N goes through the singular points h_i and h'_i, but it must also pass through two fixed points, say a and b; *i. e.*, two points in which the degenerate polar conics of h_i and h'_i meet. The tangents to N at a and b meet in a point from which a line pencil cuts out the corresponding points on N. According to (61) this point must be J_i, and we have the result:

(62) *If the points of contact of tangents from J_i to N are a and b, the polar conics of h_i and h'_i are respectively $\overline{h'_i a}\ \overline{h'_i b}$ and $\overline{h_i a}\ \overline{h_i b}$.*

To the three double tangents of R_4 correspond the parameters on N of the point pairs h_i, h'_i; and to the three double tangents of $\overline{R}_4$ correspond the parameters on N of the point-pairs k_i, k'_i. Since the tangents to N at a and b pass through J_i, a singular point of $\bar{Q}$, both correspond in $\bar{Q}$ to the singular line H_i. Therefore H_i is a double tangent of $\overline{R}_4$ with the parameters of a and b; *i. e.*, a and b are a pair k_i, k'_i on N. Since $\overline{ab}$ or $\overline{k_i k'_i}$ or K_i is the polar line of J_i, we find that

(63) *The triangle H_i of double tangents of $\overline{R}_4$ and the triangle K_i of double tangents of R_4 are polar triangles of N. The polar conic of h_i as to f is $\overline{h'_i k_i}\ \overline{h'_i k'_i}$; of k_i as to B is $\overline{k'_i h_i}\ \overline{k'_i h'_i}$.*

The following corollary may be worth noting:

(64) *Three pairs of corresponding points determine a cubic uniquely. If the three pairs lie on a conic, the polar triangle of their joins cuts out another set of three pairs. The two cubics constructed from the two sets of three pairs lie in a syzygetic pencil.*

The relation of the syzygetic pencil to the conic has been given above.

Another consequence of (63) is that the Jacobians of pairs of the three quadratics (on N) $h_i h'_i$ are $k_i k'_i$, and *vice versa*; and that the products of corresponding quadratics $h_i h'_i\, k_i k'_i$ are sets of the common covariant $I_{1,3}$ of $I_{2,2}$ and $\overline{I}_{2,2}$.

In order to define the conic N with reference to a given R_4, we recall that the tangents of R_4 from a point y are the polar lines as to B of the four points on N cut out by the polar conic of y as to B. If two of these four points coincide, y is a point of R. But the conic polar of k_1 as to B meets N at k'_1 twice, and at h_1 and h'_1. Thus k_1 is a point on the double tangent K_1 where it meets R_4 again.

(65) *The conic N passes through the six points k_i, k'_i where the double tangents of R_4 meet R_4 again; and through the six points h_i, h'_i where the double tangents of $\overline{R}_4$ meet $\overline{R}_4$ again.*

Other facts as to the mutual relation of R_4 and $\overline{R}_4$ with regard to N are readily shown. For example, the given $I_{2,2}$ has four double pairs s_i, s'_i, $i=1,2,3,4$. The four lines $\overline{s_i s'_i}$ are the fixed lines of $\overline{Q}$. The tangent of N at s_i is transformed by $\overline{Q}$ into the tangent at s'_i whence *the eight common tangents of $\overline{R}_4$ and N have, for parameters on N, the parameters (on R_4) of the eight tangents at the four double points of R_4*, etc.

But the statements (63) and (65) identify the curves R_4 and $\overline{R}_4$ with the pair of curves R and P which are treated at great length by Meyer,* though in dual form and in a quite different setting. Possibly the most interesting result obtained is the reciprocity between three pairs of corresponding points of Δ on a conic and the six further intersections of the polar conics of the six points with the conic.

§ 9. *Interpretation of Certain Invariants of the Form $I_{2,2}$.*

The rational quartic loci just obtained have been of class four. The curve of order four, $R^{(4)}$, is handled more commonly, and results will be stated for this curve even though they have been gotten in the dual form.

From the obvious properties of the $I_{2,2}$ there follows:

(66) *The curves f and Δ cut out on N the points whose parameters on N are the parameters on $R^{(4)}$ of the six flexes and the three double points respectively; i. e., the six flexes and the three double points are determined by the equations:*

$$(a_1x)^2(a_2x)^2(a_3x)^2=0$$

and

$$(\delta_1x)^2(\delta_2x)^2(\delta_3x)^2=0$$

respectively.

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* *Apolarität*, pp. 258–272.

The three cubics f, A, and B lie in the syzygetic pencil of f and Δ. The polar conics of P as to three cubics lie in a pencil whose base-points are an $A^4_{3,3}$ of each curve of the syzygetic pencil. To the ∞^2 positions of the pole P correspond ∞^2 such antorthic sets of the syzygetic pencil,* *one on each involution conic.* On N this set of four points is the common tetrad of $I_{2,2}$ and $\bar{I}_{2,2}$. Since it is the four points in which the polar conic of P (the pole of N as to A) as to f meets N, its equation is (see 47)

$$(67) \qquad (\sigma_1\sigma_2)^2(\sigma_3 a_3)^2(a_1 x)^2(a_2 x)^2 = 0.$$

According to Caporali,† if three points of $A^4_{3,3}$ lie on f, the fourth also lies on f, and the four points are the meets, other than P, of f and the polar conic as to f of a point P on f. Hence

(68) *If three flexes of the curve $R^{(4)}$ lie on a line, a fourth also lies on the line.‡ The invariant condition that four flexes lie on a line is that the pole P lie on f or, in terms of the coefficients of $I_{2,2}$, that the invariant of degree four* (see 52)

$$(\sigma_1\sigma_2)^2(a_1a_2)^2(\sigma_3a_3)^2$$

vanish. If this condition is satisfied, the four flexes are obtained from the equation (67).

In considering invariants of $I_{2,2}$ which are at the same time invariants of f, one must examine each particular case with regard to possible variations in the number of involution conics and with regard to the determination of the pole P. In this connection the list of canonical forms and their covariants tabulated by Gordan § for all special cubics f is very convenient.

$S=0$ is the condition that f have an antorthic set of three points such that any two of the three are apolar to f; *i.e.*, an $A^3_{2,3}$. Then the ∞^4 $A^4_{3,3}$'s of f are distributed, ∞^2 at a time, on the conics of the net through $A^3_{2,3}$. Let the parameters of $A^3_{2,3}$ on N, any one of the involution conics, be μ_1, μ_2, μ_3. The polar conic of the point μ_i as to f is the square of the line $\overline{\mu_k\mu_l}$. Thus, for the given value μ_i, $I_{2,2}$ reduces to an $I_{1,2}$ which consists of the neutral pair $\mu_k\mu_l$ and an arbitrary point (§6). Hence $I_{2,2}$ contains the neutral triad $\mu_1\mu_2\mu_3$ and $R^{(4)}$ has a triple point.

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* Caporali, *loc. cit.*, p. 52, § 23.

† *Loc. cit.*, p. 52, § 22.

‡ Brill, *Math. Annalen*, Vol. XII.

§ *Trans. American Math. Soc.*, Vol. I, p. 492.

(69) *The vanishing of the invariant $\sqrt{\frac{1}{6}S}$ of the second degree in the coefficients of $I_{2,2}$ is the condition that $R^{(4)}$ have a triple point.*

To express certain conditions on f, the contravariant t, the evectant of the invariant T, is necessary. Let

(70) $$t = u_t^3 = (t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2.$$

Since $\sqrt{\frac{1}{6}S}$ is rational in the coefficients of $I_{2,2}$, the discriminant of f, $R = T^2 - \frac{1}{6}S^3$, breaks up into two rational factors, $R_1 = T + S\sqrt{\frac{1}{6}S}$ and $R_2 = T - S\sqrt{\frac{1}{6}S}$. To determine the involution conics, we take f in the canonical form

$$\begin{aligned}
f &= x_1^3 + x_2^3 + 6x_1x_2x_3,\\
\Delta &= -6(x_1^3 + x_2^3 - 2x_1x_2x_3),\\
S &= 24, \quad T = 48, \quad \pm\sqrt{\tfrac{1}{6}S} = \pm 2,\\
\Delta - 2f &= -8(x_1^3 + x_2^3),\\
\Delta + 2f &= -4(x_1^3 + x_2^3 - 6x_1x_2x_3).
\end{aligned}$$

From the formula (55), or directly from the relation (36) for $\lambda = \pm 2$, two nets of involution conics are found:

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3(2x_3^2 - x_1x_2).$$
$$\sigma_1(x_1^2 - 2x_2x_3) + \sigma_2(x_2^2 - 2x_3x_1) + \sigma_3x_1x_2.$$

The conics in the second net are the polar conics of $\Delta + 2f$, each conic having a definite pole. The conics of the first net are not all polar conics, and in this case the correspondence between pole and conic is no longer unique. But from (47′) the correspondence between pole and conic fails when $R_1 = 0$. Hence $R_1 = 0$ is the condition that N be a conic of the first net, and $R_2 = 0$ is the condition that N be a conic of the second net. From the usual* parametric representation of Δ, it is easily verified that the conics of the first net cut Δ in three pairs of corresponding points, while those of the second net cut Δ in two pairs of corresponding points in addition to the self-corresponding point, the common double point of f and Δ.

* Cf. Clebsch-Lindemann, Vol. II, p. 336. In the same chapter the invariant conditions on f to be used later are found.

Let $R_2 = 0$, whence N is a conic of the second net on which the double point of f has a parameter μ_1. Since the polar conic of μ_1 as to f is a pair of lines which meet at μ_1, for the value μ_1, $I_{2,2}$ reduces to an $I_{1,2}$ which has a neutral point μ_1, (§6). Hence $I_{2,2}$ arises from a curve $R^{(4)}$ with a cusp.

(71) *The vanishing of the invariant, $T - S\sqrt{\frac{1}{6}S}$, of the sixth degree in the coefficients of $I_{2,2}$ is the condition that $R^{(4)}$ have a cusp.*

When f has a double point, the contravariant $\Pi = St - Ts$ is the cube of the double point. Since now $T = S\sqrt{\frac{1}{6}S}$, this is $S(t - \sqrt{\frac{1}{6}S}s)$, or, in the binary notation, Π is proportional to

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 - \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2.$$

Hence

(72) *When $T - S\sqrt{\frac{1}{6}S} = 0$, the parameter of the cusp is determined from the equation $(t_1x)^2(t_2x)^2(t_3x)^2 - \sqrt{\frac{1}{6}S}(\sigma_1x)^2(\sigma_2x)^2(\sigma_3x)^2 = 0$, which is a perfect sixth power.*

Let $R_1 = 0$, whence N is a conic of the first net. Evidently the cubics f and $B = \Delta + 2f$ are interchanged by the harmonic perspectivity with the double point of f, $\xi_3 = 0$, as center and the line of flexes of f, $x_3 = 0$, as axis. But this perspectivity leaves N unaltered. Hence the envelopes of class four, R_4 and $\bar{R}_4$, of the preceding paragraph are interchanged by the perspectivity, as also are their triangles of double tangents H_i and $\bar{K}_i$. On N the perspectivity is an ordinary quadratic involution which interchanges the pairs of points h_i, h_i' with k_i, k_i'. Since Π is the cube of ξ_3 and ξ_3 does not lie on N (when N is a proper conic), the polar of N as to Π is ξ_3. Π is now proportional to

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2,$$

and the polar of N as to Π is represented by the quadratic $(t_1t_2)^2(t_3\bar{x}_3)^2 + \sqrt{\frac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2$. But this quadratic determines the fixed points of the perspectivity on N.

(73) *The vanishing of the invariant, $T + S\sqrt{\frac{1}{6}S}$, of the sixth degree in the coefficients of $I_{2,2}$ is the condition that there shall exist a binary involution on $R^{(4)}$ which interchanges the parameters of each double point of $R^{(4)}$ with the parameters of*

the two tangents from the double point. The covariant $(t_1t_2)^2(t_3x)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3x)^2$ *of the sixth degree determines the fixed points of the involution.**

Let $S=0$ and $T=0$. Then f has a cusp, and there is a single net of involution conics any one of which, say N, passes through the cusp with a parameter μ_1, through the meet of the cusp tangent and inflexion tangent with a parameter μ_2, and touches at μ_1 the join of the inflexion and cusp. Since the polar conic of μ_1 as to f is the square of the line $\overline{\mu_1\mu_2}$, $I_{3,3}$ has ∞^1 tetrads consisting of μ_1 twice, μ_2, and an arbitrary point.

(74) *The vanishing of the invariants,* $\sqrt{\tfrac{1}{6}S}$ *and* T, *of degree two and six is the condition that* $R^{(4)}$ *have a triple point with two coincident branches.*

Let f have two double points. Then

$$f = x_1^3 + 6x_1x_2x_3, \qquad \Delta = -6(x_1^3 - 2x_1x_2x_4),$$
$$S = 24, \qquad T = 48, \qquad \pm\sqrt{\tfrac{1}{6}S} = \pm 2,$$
$$\Delta - 2f = -8x_1^3, \qquad \Delta + 2f = -4(x_1^3 - 6x_1x_2x_3).$$

Corresponding to the values $\lambda = -2$ and $\lambda = +2$, there are respectively the two nets of involution conics

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2,$$
$$\sigma_1(x_1^2 - 2x_2x_3) + \sigma_2x_3x_1 + \sigma_3x_1x_2.$$

As before, the two nets are distinguished by the vanishing of R_1 and R_2 respectively. When $R_2 = 0$, the conic N passes through the two double points and the curve $R^{(4)}$ has two cusps. The two double points are cut out on N by $\Delta - 2f$ three times. The contravariant Π vanishes identically. Hence

(75) *When* $T - S\sqrt{\tfrac{1}{6}S} = 0$ *and*

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 - \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2 \equiv 0,$$

the curve $R^{(4)}$ *has two cusps which are determined by the equation with two triple roots*

$$(\delta_1x)^2(\delta_2x)^2(\delta_3x)^2 + \sqrt{\tfrac{1}{6}S}(a_1x)^2(a_2x)^2(a_3x)^2 = 0.$$

* We may note here that the theorem of Meyer (*Apolarität*, p. 260, footnote): "Es giebt eine bestimmte projectivische Beziehung, die irgend drei Elementenpaare in ihre bezüglichen Funktionaldeterminanten überführt" is not correct. Another version of the statement is:—Two perspective triangles (*i. e.*, two polar triangles as to a conic) can be interchanged by a collineation which is necessarily involutory;—and this is true only when corresponding vertices of the triangles are harmonically separated by the center and axis of perspection. This is one condition on the two perspective triangles or one condition on one triangle and the conic. Hence the three quadratics in the theorem quoted must be subject to one condition.

The error is the result of another in the statement of the theorem, (π), p. 260. Only when the invariant $T + S\sqrt{\tfrac{1}{6}S}$ of the $I_{3,3}$ of either curve R and P vanishes is the theorem (π) correct.

On the other hand, let $R_1 = 0$, whence N is a conic of the first net unaltered by the three harmonic perspectivities whose centers and axes are (ξ_1, x_1), (ξ_2, x_2), and (ξ_3, x_3). The last two interchange f and B, while the first leaves each unaltered. The two envelopes, R_4 and $\overline{R}_4$, are again mutually related, each being unaltered by one perspectivity and the two being interchanged by the other two perspectivities. N cuts the common line, $x_1 = 0$, of f and Δ in two points with parameters μ_1 and μ_2 and cuts Δ further in two pairs of points, each pair being harmonically separated by $x_1 = 0$ and $\xi_1 = 0$. Hence the perspectivity (ξ_1, x_1) leaves the three double tangents of R_4 (as well as of $\overline{R}_4$) each unaltered, the points of contact on one double tangent being unaltered, while those on the other two are interchanged. Since $\Delta - 2f = -8x_1^3$, the polar of N in lines as to $\Delta - 2f$ is the line $x_1 = 0$. Hence

(76) *When* $T + S\sqrt{{}^1S} = 0$ *and*

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_1\bar{x}_2)^2(\sigma_3\bar{x}_3)^2 \equiv 0,$$

the curve $R^{(4)}$ is unaltered by a harmonic perspectivity with center at one double point and axis through the other two. The first double point is also a double flex-point. The parameters of the double flex-point are determined by the equation

$$(\delta_1\delta_2)^2(\delta_3 x)^2 + \sqrt{\tfrac{1}{6}S}(a_1a_2)^2(a_3x)^2 = 0.$$

There exist on the curve two binary involutions which interchange the parameters at the double flex-point, and, at each of the other double points, interchange the parameters of the double point with the parameters of tangents from the double point.

Let f have three double points. Then

$$f = 6x_1x_2x_3, \qquad \Delta = 12x_1x_2x_3, \qquad S = 24, \qquad T = 48,$$
$$\pm\sqrt{\tfrac{1}{6}S} = \pm 2, \qquad \Delta - 2f \equiv 0, \qquad \Delta + 2f = 24x_1x_2x_3.$$

The involution conics are again in two nets,

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2,$$
$$\sigma_1x_2x_3 + \sigma_2x_3x_1 + \sigma_3x_1x_2,$$

corresponding respectively to $R_1 = 0$ and $R_2 = 0$. For conics of the second net $R^{(4)}$ has three cusps; for those of the first net, since f and Δ are proportional, the double points coincide with the flexes. The invariant condition in the ternary domain is $\Delta - 2f \equiv 0$.

(77) *When* $T - S\sqrt{\frac{1}{6}S} = 0$ *and*

$$(\delta_1 x_1)^2(\delta_2 x_2)^2(\delta_3 x_3)^2 + \sqrt{\tfrac{1}{6}S}(a_1 x_1)^2(a_2 x_2)^2(a_3 x_3)^2 \equiv 0,$$

the curve $R^{(4)}$ *has three cusps; when, however,* $T + S\sqrt{\frac{1}{6}S} = 0$ *and the same covariant vanishes, the curve* $R^{(4)}$ *has three double flex-points.*

Let f be a conic and one of its tangents.

$$f = 3x_1(x_2^2 + x_3 x_1), \qquad \Delta = -6x_1^3, \qquad S = T = 0, \qquad t \equiv 0.$$

The involution conics are now in a single net,

$$\rho_1 x_1^2 + \rho_2 x_1 x_2 + \rho_3(x_2^2 - x_3 x_1),$$

whose members touch the line of f, $x_1 = 0$, at the point, $\xi_3 = 0$, where it touches the conic of f. Hence N meets Δ at this point only with a parameter μ_1. The polar conic of μ_1 as to f is the square of a line which touches N at μ_1. For the value μ_1, $I_{2,2}$ reduces to an $I_{1,2}$ consisting of μ_1 twice and an arbitrary point. Thus the curve $R^{(4)}$ has a triple point which is formed by the coalescence of two cusps and a node into a smooth point. Such a singularity, the reciprocal of an undulation point, will be called a bi-stationary point.

(78) *The identical vanishing of* $(t_1 x_1)^2(t_2 x_2)^2(t_3 x_3)^2$ *is the condition that* $R^{(4)}$ *have a bi-stationary point. The parameter of the point is determined as the six-fold root of the equation* $(\delta_1 x)^2(\delta_2 x)^2(\delta_3 x)^2 = 0$.

For further degenerations of the cubic f into three lines the involution conics are also all degenerate and the above enumeration of types of $R^{(4)}$ which correspond to types of the cubic f is complete. But for every type of f there will exist sub-types of $R^{(4)}$ which correspond to special positions of the pole P of N. Without attempting a complete discussion of these cases, some of especial interest may be pointed out.

If the polar conic of P as to A touches Δ at one point, it must touch Δ at the corresponding point also, and on $R^{(4)}$ two double points come together to form a tac-node. Since the polar conics of A which touch Δ form a singly infinite system, their poles lie on a locus Γ which we shall determine. Γ is also the envelope of the polar lines as to A of points on Δ. Such an envelope is in general of the sixth class, but since the polar lines of corresponding points on Δ are the same, the class of Γ is three. Δ and A are each unaltered by the collineation G_{18} of the syzygetic pencil; Γ is then unaltered by the same group and can be assumed as

$$u_1^3 + u_2^3 + u_3^3 + 6\gamma u_1 u_2 u_3.$$

Here γ is determined by requiring the locus Γ to touch the polar line as to $A=\Sigma x_1^3+6ax_1x_2x_3$ of the point $0, 1, -1$, a flex-point on Δ. This polar line is $-2ax_1+x_2+x_3=0$, whence $-8a^3+2-12a\gamma=0$. If $f=\Sigma x_1^3+6mx_1x_2x_3$, then a satisfies the relation, $4ma^2+4m^2a+1=0$. Lowering, by means of this relation, powers of a higher than the first, the condition on γ becomes $-4m^3+1-6m\gamma=0$, whence $\gamma=\dfrac{1-4m^3}{6m}$ and Γ is the common Cayleyan of f, A, and B.

(79) *If s is the common Cayleyan of three cubics, the polar conic of a point P on s as to one of the cubics touches each of the Hessians of the other two cubics at a pair of corresponding points.*

If the polar conic of P as to A passes through a common point of f and Δ, *i. e*, if P lies on a flex-tangent of A, then $R^{(4)}$ has a flex-point at a double point. Hence

(80). *The invariant condition on the coefficients of $I_{2,2}$ in order that $R^{(4)}$ have a tac-node is that the pole $P=(\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2$ shall lie on the Cayleyan $(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2$; the condition that $R^{(4)}$ have a flex at a double point is that the pole P shall lie on one of the nine flex-tangents of the cubic*

$$(\delta_1x_1)^2(\delta_2x_2)^2(\delta_3x_3)^2+\sqrt{\tfrac{1}{3}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2.$$

These invariant conditions are not given in symbolic form, because a direct substitution of the coordinates of P in the locus in question gives the required invariant multiplied by extraneous invariants.* Until more is known of the relations among the invariants of the form $I_{2,2}$, explicit expressions are out of the question. Take, for instance, the condition that $R^{(4)}$ have an undulation point, two coincident flex-points. Then $N=(x_1x_2)^2$ must touch f. The tact-invariant of the conic and the cubic is of degree ten in the coefficients of the cubic. Since the undulation condition is known to be of degree four, an invariant of degree six must factor out of the tact-invariant. This extraneous factor must be R_2, since for $R_2=0$, N touches f (*i. e.*, passes through the double point of f), and on $R^{(4)}$ two flexes coincide (at the cusp of R_4).

The tac-node condition, obtained by substitution of the pole P, as in (80), or by forming the tact-invariant of N and Δ, appears in either case as an invariant of degree thirty. Since two contacts with Δ are involved simultaneously,

* *E. g.*, the condition that P lie on a curve of the syzygetic pencil is of degree twelve when obtained by substitution, but contains the factor R_1. See (52).

this is probably a perfect square from which R_1 also factors, leaving an invariant of degree nine as the probable true tac-node condition.

The true undulation-condition of degree four in the coefficients of $I_{2,2}$ can be obtained as follows: On $R^{(4)}$ the $I^4_{2,2}$ determines the apolar involution $I^4_{1,3}$, the so-called "fundamental involution on $R^{(4)}$." $I^4_{1,3}$ is a linear covariant of $I^4_{2,2}$. It is, in fact,

$$(r_1t_1)^3(r_2t_2)^3 \equiv 2(a_1t_1)(a_2t_1)(a_3t_1)(a_1t_2)(a_2t_2)(a_3t_2) - (t_1t_2)^2(a_1a_2)^2(a_3t_1)(a_3t_2)$$

if t_1 and t_2 belong to a tetrad of $I^4_{1,3}$. This form has been represented as a quadric in S_3, the vanishing of whose discriminant [given symbolically in (16)] is the condition that all the tetrads of $I^4_{1,3}$ have a common point which is an undulation point on $R^{(4)}$.

Finally, it may be worth while to give the point-equations of the class quartics, R_4 and $\overline{R}_4$. Since R_4 is the envelope of the polar lines of points on N as to B, it is also the locus of points, y, whose polar conics as to B touch N. The invariants of the two conics, $N = A_PA^2_x$ and $B_yB^2_x$, are:

$$\begin{aligned}(81)\qquad A_{111} &= A_PA'_PA''_P(AA'A'')^2 = -\tfrac{1}{3}R_1[\Delta^3_P - 3\sqrt{\tfrac{1}{6}S}f^3_P],\\ A_{112} &= A_PA'_PB_y(AA'B)^2 \quad = -\tfrac{1}{3}R_1[B^2_PB_y],\\ A_{122} &= A_PB_yB'_y(ABB')^2 \quad = -\tfrac{1}{3}R_2[A_PA^2_y],\\ A_{222} &= B_yB'_yB''_y(BB'B'')^2 = -\tfrac{1}{3}R_2[\Delta^3_y - 3\sqrt{\tfrac{1}{6}S}f^3_y].\end{aligned}$$

To obtain $\overline{R}_4$, f replaces B and the conic $a_ya^2_x$ replaces $B_yB^2_x$. The invariants of N and $a_ya^2_x$ are:

$$\begin{aligned}(82)\qquad \overline{A}_{111} &= A_{111} = -\tfrac{1}{3}R_1[\Delta^3_P - 3\sqrt{\tfrac{1}{6}S}f^3_P],\\ \overline{A}_{112} &= \qquad\quad \tfrac{1}{3}R_1[a^2_Pa_y],\\ \overline{A}_{122} &= \qquad\quad \sqrt{\tfrac{1}{6}S}[A_PA^2_y],\\ \overline{A}_{222} &= \qquad\quad \Delta^3_y.\end{aligned}$$

In terms of these, the tact-invariant of N and $B_yB^2_x$ is*

$$4(A_{111}A_{122} - A^2_{112})(A_{112}A_{222} - A^2_{122}) - (A_{111}A_{222} - A_{112}A_{122})^2 = 0,$$

which is the sextic point-equation in variables y of the quartic R_4.

The locus of points y from which the tangents to R_4 are equianharmonic is

$$2i = 3(A^2_{112} - A_{122}A_{111}) = 0,$$

which is the conic through the cusps of R_4. Since A_{122} is the conic N, we find

* Clebsch-Lindemann, Vol. I, p. 371 et seq.

dually that the conic which touches the six flex-tangent of the point-quartic $R^{(4)}$, and the conic which touches the tangents from the three double points of $R^{(4)}$ *touch at two points.*

The locus of points y from which the tangents to R_4 are harmonic is the cubic

$$8/3j = 3A_{111}A_{112}A_{122} - A_{211}^2A_{222} - 2A_{112}^3 = 0.$$

From these formulae the locus of points y from which the tangents to R_4 have any desired anharmonic ratio (a sextic curve) can be gotten.

The point-equation of R_4 is symmetrical in y and B, and P and A. The locus of points for which R_4 passes through P is a curve of order twelve, the product of the four flex-triangles. Thus the condition that the pole P lie on a flex-triangle is the discriminant of the quartic covariant of (67) and is, at least formally, of degree twenty-four. There seems to be no simple geometrical peculiarity of $R^{(4)}$ which corresponds to this invariant.

In an article to appear in the next number of this Journal, the relation between the rational space quintic curve and the plane quartic curve (subject to one condition), and the relation between the rational plane quintic curve and the cubic surface in S_3 will be discussed.

BALTIMORE, *December* 31, 1908.

The Birational Transformations of Algebraic Curves of Genus Four.

BY ANNA L. VAN BENSCHOTEN.

In this paper the groups of birational transformations which leave curves of genus 4 invariant are obtained, and some geometrical properties connected with such transformations are considered. This has been done for the space sextic situated upon an hyperboloid by Wiman,* who has also outlined in the same paper the groups for plane curves derived from those sextics which lie upon a cone.

The discussion naturally falls into three main divisions: first, the hyperboloidal case, which deals with binodal quintics and a few sextics which are the projections of a space sextic lying upon a hyperboloid; second, the conical case, wherein the corresponding normal curve is on a cone; and, third, the hyperelliptic case.

§ 1. *The Hyperboloidal Case.*

1. As is well known, the normal form for every plane curve of genus 4 which is not hyperelliptic is a curve of the fifth order. That is, all such curves $C_m(4)$, whatever be the degree m, possess a point-group series g_5^2 and are thus birationally equivalent to a $C_5(4)$, a quintic with two nodes.

When the nodes are distinct, the triangle of reference OIJ can be selected with vertices I and J or $(0, 0, 1)$ and $(1, 0, 0)$ at the nodes. Then the triply infinite system of adjoint conics is written

$$axy + by^2 + cyz + dxz = 0.$$

Whence putting for xy, y^2, yz, and xz the new variables $\rho x'$, $\rho y'$, $\rho z'$, and $\rho w'$, the result is a plane section of the hyperboloid F_2 whose equation is

$$x'z' = y'w'.$$

* A. Wiman: Ueber die algebraischen Curven von den Geschlechtern $p=4$, 5, und 6, welche eindeutige Transformationen in sich besitzen, *Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar.* Stockholm, 1895–96.

The substitution in the quintic gives a cubic surface F_3 whose intersection with F_2 is a space sextic S_6.

Whenever, by a birational transformation, the plane quintic C_5 goes into itself, the corresponding space sextic S_6 must also go into itself, and conversely. The adjoint conics ϕ_2 merely interchange, as do also the corresponding plane sections of F_2. Hence the transformations in space are all linear. Since $p=4$, every generator of F_2 is a trisecant. The inflexional tangents of S_6 must be generators of F_2, for no other straight lines have three points in common with the curve. S_6 can have no point-singularity; for if projected from a double point or cusp it would give a quartic whose genus is less than four.

Collineations which leave S_6 invariant, leave F_2 invariant. The linear transformations of F_2 are of two kinds: first, those in which the systems of generators interchange; second, those in which the systems do not interchange, though the generators of either or of both systems may interchange among themselves. It is then convenient to have coördinates that distinguish between the systems of generators. This is done by the following equations:

$$x'/w' = y'/z' = y_1/y_2 \text{ and } y'/x' = z'/w' = x_1/x_2,$$

which represent two pairs of planes intersecting respectively on F_2 in lines which intersect and thus belong to different systems. Then x_1, x_2 and y_1, y_2 may be regarded as coördinates of the two systems.

A cubic in both sets x_1/x_2, y_1/y_2, viz.,

$$\theta_3(x_1/x_2;\ y_1/y_2) = 0,$$

will then determine three generators of one system for any particular generator of the other, and therefore define a sextic curve whose trisecants are generators of the hyperboloid.

Such a sextic, if projected* from a point on the curve, goes into a plane quintic with two distinct nodes; from a point on F_2 but not on the curve, a sextic with two triple points; from a point not on F_2, a sextic with six nodes lying on a conic, unless the center of projection be taken at the vertex of a cubic cone on which S_6 lies, in which case the plane curve is manifestly a cubic counted twice.

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*Clebsch: Vorlesungen über Geometrie II, S. 414.

If the center of projection be taken at the vertex D or $(0, 0, 0, 1)$ of the reference tetraedron, the substitutions are

$$\begin{pmatrix} x', & y', & z', & w' \\ xy, & y^2, & yz, & xz \end{pmatrix} \text{ or } \begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y, & x, & y, & z \end{pmatrix},$$

where the variables in the lower row are to replace those in the upper.

Collineations of the first kind wherein the generator systems interchange may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ay_1+by_2, & cy_1+dy_2, & ex_1+fx_2, & gx_1+hx_2 \end{pmatrix},$$

which, if generators through two invariant points, say $x_1=y_1=0$ and $x_2=y_2=0$, be taken as edges of the reference tetraedron, may be reduced to the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ b_1y_1, & b_2y_2, & a_1x_1, & a_2x_2 \end{pmatrix}.$$

Collineations of the second kind may leave the generators of one system severally invariant while interchanging those of the other system, or they may interchange generators of both systems among themselves. Such are

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & bx_1, & y_1, & y_2 \end{pmatrix} \text{ and } \begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & bx_1, & cy_2, & dy_1 \end{pmatrix}.$$

The product of two collineations of the first kind is evidently either identity or one of the second kind. When therefore an equation is invariant under an operation T of the first kind, it is likewise invariant under T^2, which is an operation of the second kind if the period of T exceeds two. Thus the group which contains operations interchanging the systems of generators contains as many which leave the systems invariant. When the systems do not interchange, the groups of collineations are simply isomorphic with the well-known binary linear groups. Hence there are no new simple groups.

2. If the S_6 can be transformed into itself by a collineation of period 2, there are two cases arising according as the transformation is of the first or of the second kind. First suppose that the curve belongs to a group G_2 of order 2 under which the generator systems interchange. The collineation may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_1, & y_2, & x_1, & x_2 \end{pmatrix},$$

or, in space coördinates,

$$\begin{pmatrix} x', & y', & z', & w' \\ z', & y', & x', & w' \end{pmatrix},$$

which, when projected from the vertex (0, 0, 0, 1), becomes, for the plane,

$$\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix}.$$

Hereafter accents will be omitted from the space coördinates.

It is evident that the space collineation is a central perspective whose center $x+z=y=w=0$ is the pole as to F_2 of the plane of perspective $x-z=0$.

The reference tetraedron can be so chosen that two vertices D or $x_1=y_1=0$ and B or $x_2=y_2=0$ are on the curve. Then $a_0=d_3=0$ in the equation

$$x_2^3y_2^3\theta_3(x_1/x_2,\, y_1/y_2) \equiv x_1^3\sum_{n=0}^{3} a_n y_1^{3-n}y_2^n + x_1^2x_2\sum_{n=0}^{3} b_n y_1^{3-n}y_2^n + x_1x_2^2\sum_{n=0}^{3} c_n y_1^{3-n}y_2^n + x_2^3\sum_{n=0}^{3} d_n y_1^{3-n}y_2^n = 0. \quad \text{(I)}$$

In order that this equation be invariant under the above transformation, the following conditions must be fulfilled:

$$a_1=b_0,\ a_2=c_0,\ a_3=d_0,\ b_2=c_1,\ b_3=d_1,\ c_3=d_2.$$

Then (I) reduces to

$$a_3(x_1^3y_2^3+x_2^3y_1^3)+(a_2x_1y_1+b_3x_2y_2)(x_1^2y_2^2+x_2^2y_1^2) + (a_1x_1^2y_1^2+b_2x_1x_2y_1y_2+c_3x_2^2y_2^2)(x_1y_2+x_2y_1)+x_1x_2y_1y_2(b_1x_1y_1+c_2x_2y_2)=0,$$

or, in space coördinates,

$$a_3(x^3+z^3)+(a_2y+b_3w)(x^2+z^2)+(a_1y^2+b_2yw+c_3w^2)(x+z)+yw(b_1y+c_2w)=0.$$

This cubic surface and F_2 intersect in the sextic curve. It projects from D into the plane quintic

$$a_3y^2(x^3+z^3)+y(a_2y^2+b_3xz)(x^2+z^2)+(a_1y^4+b_2xy^2z+c_3x^2z^2)(x+z) + xyz(b_1y^2+c_2xz)=0.$$

The first two of these coefficients can be reduced to unity by change of scale. The quintic is then written

$$y^2(x^3+z^3)+y(y^2+axz)(x^2+z^2)+(by^4+cxy^2z+dx^2z^2)(x+z) + xyz(ey^2+fyz)=0, \quad (1)$$

with the transformation

$$H\equiv\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix},$$

which transformation is unaltered by the above change of scale.

3. As was observed before, the space collineation for (1) is a central involution with center V at $(1, 0, -1, 0)$. The center is not on the hyperboloid and every line through it cuts the hyperboloid twice. Then all lines to points

of S_6 from V are bisecants, generators of a cubic cone K_3. The cone is clearly of order 3, for any plane through V meets S_6 in six points which must lie in pairs on three bisecants. The points $P_i\,(i=1, 2, \ldots, 6)$ in the plane of perspective $x-z=0$ go into themselves by the transformation of period 2, hence the bisecants VP_i are tangent to S_6 at P_i. Two of the points P_i are at the vertices (0, 1, 0, 0) and (0, 0, 0, 1) of the reference tetraedron). The cubic cone has nine inflexional generators which lie by threes in twelve planes. An inflexional tangent plane to K_3 contains but two distinct points of S_6. These are interchanged by the transformation of period 2. Hence such a plane is a double-osculating plane. When VP_i is an inflexional generator, the corresponding inflexional tangent plane is a sextactic plane. In any case an osculating plane at P_i contains at least four consecutive points of S_6. For the osculating plane contains three consecutive points of S_6, hence two consecutive bisecants, on each of which there is a pair of consecutive points. Such a plane is therefore stationary.

The quintic is derived from the sextic by projection from the vertex D of the reference tetraedron $ABCD$. Let lines from D to A, B, C, V pierce the plane of projection in J, O, I, D' respectively. Then D', the image of D, is on the quintic and is the center of the plane homology H. A tangent at D' must have either three- or five-point contact. The five lines from D' to the points in which C_5 intersects the axis of homology are tangent to the curve at those points. The nodes at I and J are interchanged by H and are both of the same kind, crunodes, acnodes or cusps according as a^2-4d is positive, negative or zero. Unless they are cusps, the class of the curve is 16. So the five ordinary tangents from D' and the inflexional tangent at D' leave in general four double tangents from D'. In the case when the tangent at D' has contact of the fourth order, there will be but three double tangents. The double tangents are the images of the planes tangent to K_3 which intersect K_3 in the generator VD. There will be four such planes except when VD is an inflexional generator. If, however, one of them is tangent also to F_2, the plane curve is bicuspidal and the bitangents from D' are images of the other three tangent planes.

The nine inflexional tangent planes to K_3 give as images nine adjoint conics each having double three-point contact with C_5. The images of the eighteen points of intersection of F_2 with the nine inflexional generators of K_3 lie by sixes on twelve adjoint conics, because the cross-sections of K_3 are non-singular cubics whose points of inflexion lie by threes on twelve lines, and a plane containing

two inflexional generators must therefore contain a third. So an adjoint conic containing four of the eighteen three-point contact points must contain six.

Moreover, since the osculating planes of S_6 at the six points in the plane of perspective have four-point contact, their images, adjoint conics, will have four-point contact with C_5. The image, however, of the osculating plane at D breaks up into the y-axis and a line through D' tangent to the curve. As the y-axis intersects C_5 in D', the tangent at D' must have three-point contact. This is another proof for the inflexion at D'. When the osculating plane at D has six-point contact, its image, a line through D', will of course have five-point contact.

Only four bitangents and one inflexional tangent are in general accounted for. The rest, sixty-four and thirty-two respectively, meet in pairs on the axis $x - z = 0$.

Any plane through V meets S_6 in six points collinear in pairs with V. Such a plane section of F_2 goes into an adjoint conic ϕ_2 which cuts out a point-group, G_6, the points of which lie in pairs on three lines through D'.

If the projection had been made from A or C instead of D, a sextic would have been obtained remaining invariant under a quadratic inversion.

4. The collineation of period 2 of the second kind may be put in the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & -x_2, & -y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & -y, & z, & -w \end{pmatrix}$$

which projects from D into the homology

$$T \equiv \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}.$$

Beginning as before with equation (I), it is clear that in all terms the sum of the exponents of x_2 and y_1 must be odd or even in order that the above transformation may leave the equation unaltered. Thus either

$$a_0 = a_2 = b_1 = b_3 = c_0 = c_2 = d_1 = d_3 = 0,$$

or

$$a_1 = a_3 = b_0 = b_2 = c_1 = c_3 = d_0 = d_2 = 0.$$

Either set of conditions gives essentially the same quintic. Selecting the former set, the equation is

$$a_1 x_1^3 y_1^2 y_2 + a_3 x_1^3 y_2^3 + b_0 x_1^2 x_2 y_1^3 + b_2 x_1^2 x_2 y_1 y_2^2 + c_1 x_1 x_2^2 y_1^2 y_2 + c_3 x_1 x_2^2 y_2^3 \\ + d_0 x_2^3 y_1^3 + d_2 x_2^3 y_1 y_2^2 = 0,$$

or, in space coördinates,

$$d_0 x^3 + a_3 z^3 + xz(c_1 x + b_2 z) + x(b_0 y^2 + d_2 w^2) + z(a_1 y^2 + c_3 w^2) = 0.$$

By change of scale of x, y and z the coefficients b_0, d_0 and d_2 can be absorbed and the equation be written

$$x^3 + xz(ax + bz) + cz^3 + z(dw^2 + ey^2) + x(y^2 + w^2) = 0,$$

which gives, for the plane quintic,

$$x^3(y^2 + z^2) + xy^2z(ax + bz) + z^3(cy^2 + dx^2) + y^4(x + ez) = 0. \qquad (2)$$

The collineation in space is an axial involution. That is, under it $y = w = 0$ and $x = z = 0$ are not only invariant as a whole but all their points are invariant, because any plane through either line goes into itself. Hence a line joining two corresponding points of the S_6 must intersect these two axes and S_6 lies on a ruled surface with the two invariant lines as directrices. The directrix $y = w = 0$ does not intersect S_6, but any plane through it cuts S_6 in six points which lie on three bisecants through the point P, where the second directrix $x = z = 0$ cuts the above plane. Thus $x = z = 0$ is a triple directrix. Similarly $y = w = 0$ is a double directrix, for $x = z = 0$ contains two points of S_6, leaving four points in any plane through it to lie in pairs on two bisecants which meet in a point on $y = w = 0$. So the curve lies on an R_5, the order 5 being the sum of the multiplicities of the straight line directrices. From a point on a bisecant there are five other bisecants, which generate a ruled surface on which the given bisecant is of order 5. Any plane containing a bisecant cuts S_6 in four points besides the two on the bisecant. These four points are joined by six bisecants. Therefore the complete plane section of the surface is these six lines and the given bisecant which counts as five. The order of the ruled surface is thus 11. As R_{11} has a factor R_5, there is also an R_6.

It is important to bear in mind that the curve belonging to a G_2 of the first kind lies on a cubic cone, while the one whose group G_2 is of the second kind is on a particular ruled surface of order 5. Under special conditions the ruled surface of order 6 may break up, as will presently appear. The curve lies on a K_3 when it has a central involution, on an R_5 when the involution is axial.

5. The quintic (2) in the form obtained has at least one acnode. The center of the plane perspective is at O or $(0, 1, 0)$, which point is an inflexion. The inflexional tangent is $x + ez = 0$. This has contact of the fourth order when $ae^2 + c = e^3 + be$. The quintic cuts the y-axis in a point D', the image of D on S_6, and the tangent to C_5 at D' is $x + dz = 0$. The curve has six double tangents from O, which, with the tangent at D' and the inflexional tangent at O, make up the number 16, the class of the curve.

When $d=e$, in equation (2), the curve

$$x^3(y^2+z^2)+xy^2z(ax+bz)+z^3(cy^2+dx^2)+y^4(dz+x)=0 \qquad (3)$$

possesses, in addition to the linear transformation T, the quadric transformation

$$Q\equiv\begin{pmatrix}x, & y, & z\\ xy, & xz, & yz\end{pmatrix},$$

and thus QT, which is also quadric. The two quadric transformations correspond to two central collineations in space; and T, the product of the two quadric transformations, corresponds, as we have seen, to an axial involution. The central collineations project into quadric transformations because the center of projection is not an invariant point of either collineation.

If the center of projection were taken at A, the result instead of the quintic (2) would have been the sextic

$$y^3w^3+yz^2w(ayw+bz^2)+cz^6+z^4(dw^2+ey^2)+yz^2w(y^2+w^2)=0 \qquad (2')^*$$

whose transformation

$$T'\equiv\begin{pmatrix}y, & z, & w\\ y, & -z, & w\end{pmatrix}$$

corresponds to the above T, and to the axial space collineation. Moreover, when $d=e$ the sextic (3′) is obtained with collineations Q' and $Q'T'$ or

$$\begin{pmatrix}y, & z, & w\\ w, & \pm z, & y\end{pmatrix},$$

which correspond to the central space collineations.

The sextic (2′) has six bitangents from A', the point on IJ which is the image of A, and the sextic (3′) has in addition to the six from A', nine bitangents from O.

The condition $d=e$ for (3) might have been obtained in two ways: either by working for the S_6 which allows interchange of the systems of generators as well as the collineation for (2), or by finding the condition that the R_6 for the space sextic of equation (2) breaks up into two K_3. The latter method is as follows: the plane BCD or $x=0$ of the reference tetraedron contains two trisecants BC and CD which are cut by S_6 in $(0,1,0,0)$, $(0,\sqrt{c},\pm i\sqrt{e},0)$ and $(0,0,0,1)$, $(0,0,\pm i\sqrt{d},\sqrt{c})$ respectively. Likewise $z=0$ contains the points $(1,0,0,\pm i)$ and $(1\pm i,0,0)$. If the two sides of the quadrilateral formed by bisecants in the x-plane which meet on $x=z=0$ are met at the same

* In this section, the plane sextics are given accented numbers, and corresponding quintics unaccented.

point by those of the quadrilateral in the z-plane, R_6 breaks up. The condition that a bisecant in the x-plane meets a corresponding one in the z-plane is that certain four of the above points lie in the same plane, which condition is satisfied only if

$$\begin{vmatrix} 1, & i, & 0, & 0 \\ 1, & 0, & 0, & i \\ 0, & 0, & i\sqrt{d}, & \sqrt{c} \\ 0, & \sqrt{c}, & i\sqrt{e}, & 0 \end{vmatrix} = 0;$$

that is, if $\sqrt{c}(\sqrt{d} - \sqrt{e}) = 0$. When $c = 0$, four points in the plane $x = 0$ coincide at C and there would be no determinant. When, however, $d = e$, there are two central involutions in addition to the axial involution of (2), and S_6 is on two K_3 into which R_6 has broken up. When $c = 0$, it is observed that although there is no new transformation the curve has a stationary plane at C.

The centers V_1, V_2, or $(0, 1, 0, \pm 1)$, are on the line $x = z = 0$ and they project from D into O of the reference triangle OIJ. The plane $x + dz = 0$ is a double osculating plane at $(0, 0, 0, 1)$ and $(0, 1, 0, 0)$ and is an inflexional tangent plane to both cones. Its image is a double tangent with contact of the second order at O and ordinary contact on the z-axis. This inflexional double tangent and the six ordinary double tangents make up the number 16, the class of the curve. The eight other inflexional tangent planes to the two cones have as images adjoint conics, each having double contact of the second order with C_5.

6. A four-group can also be obtained whose operations are all of the second kind or axial. Besides the collineation belonging to (2) there may be two others of the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \pm x_1, & \pm y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & \pm w, & x, & \pm y \end{pmatrix},$$

or, for the plane,

$$\begin{pmatrix} x, & y, & z \\ yz, & \pm xz, & xy \end{pmatrix}.$$

Beginning with the form for (2) before the coefficients were absorbed by change of scale, and obtaining the condition that these last transformations will leave the equation invariant, the result is

$$x^3 + z^3 + axz(x + z) + b(xy^2 + zw^2) + c(xw^2 + y^2z) = 0,$$

which gives the plane quintic

$$y^2(x^3 + z^3) + axy^2z(x + z) + y^4(bx + cz) + x^2z^2(bz + cx) = 0. \qquad (4)$$

The lines $x \pm z = 0$ and the adjoint conics $xz \pm y^2 = 0$ are invariant under all the plane transformations of the group. They are the projections of sections of F_2 made by invariant planes $y \pm w = 0$ and $x \pm z = 0$. Each of the invariant conics intersects C_5 in six points collinear in pairs with O. The conics $xy \pm z^2 = 0$ and $yz \pm x^2 = 0$ are each invariant under the corresponding quadric transformations of the group. They are the projections of the intersections of F_2 with the cones $xy \pm z^2 = 0$ and $yz \pm x^2 = 0$. The cone $xy + z^2 = 0$ is transformed by $\begin{pmatrix} x, & y, & z, & w \\ z, & w, & x, & y \end{pmatrix}$ into $zw + x^2 = 0$. If from the last equation and that of F_2, w be eliminated, the result is $xy + z^2 = 0$. The cone $xy + z^2 = 0$ has the line $y = z = 0$ in common with the hyperboloid, and it goes into $x = w = 0$ by the above transformation. The cone and the hyperboloid have also a cubic curve in common which remains invariant under the above transformation and is projected from D into the conic invariant under the corresponding plane transformation.

The quadric transformations for C_5 may be regarded as the products of inversions by harmonic homologies. Thus:

$$\begin{pmatrix} x, & y, & z \\ yz, & xz, & xy \end{pmatrix} = \begin{pmatrix} x, & y, & z \\ xy, & xz, & yz \end{pmatrix}\begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix} = \begin{pmatrix} x, & y, & z \\ xz, & yz, & xy \end{pmatrix}\begin{pmatrix} x, & y, & z \\ y, & x, & z \end{pmatrix}, \text{ etc.}$$

The quintic (4) has an inflexion at O, but the tangent from O to the point of intersection of C_5 with the y-axis is not, as in (3), the inflexional tangent at O, but is transformed into it by the quadric transformations. The former tangent is $cx + bz = 0$, while its image, the inflexional tangent, is $bx + cz = 0$.

7. There are then two non-cyclic groups of order 4. The question arises whether there can be a cyclic G_4. In a collineation of period 4 the systems of generators must interchange, for otherwise there would be fixed generators on which the points of S_6 would have to change cyclically, which could only be when the period is 3; or else the twelve points on the edges of the reference tetraedron must be situated in groups at the vertices, which latter case occurs, as will be seen, presently in the collineation of period 5.

A collineation of period 4 can be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ iy_1, & y_2, & x_1, & -ix_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & iy, & x, & -iw \end{pmatrix},$$

or, in the plane,

$$\begin{pmatrix} x, & y, & z \\ z, & iy, & x \end{pmatrix}.$$

As the square of this collineation is the G_2 of the second equation, it is well to begin with the form for that equation before the coefficients were absorbed. With the conditions that the above transformation imposes, it is found that

$$x_1^3y_2^3 + x_2^3y_1^3 + ax_1x_2y_1y_2(x_1y_2 + x_2y_1) + (bx_1^2y_1^2 + cx_2^2y_2^2)(x_1y_2 - x_2y_1) = 0,$$

which possesses not only the above transformation but also

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \sqrt{c}x_2, & \sqrt{-b}x_1, & \sqrt{c}y_2, & \sqrt{-b}y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \sqrt{-bc}z, & cw, & \sqrt{-bc}x, & by \end{pmatrix},$$

which becomes, in the plane,

$$\begin{pmatrix} x, & y, & z \\ \sqrt{-b}yz, & \sqrt{c}xz, & \sqrt{-b}xy \end{pmatrix}.$$

From the above equation is derived the plane quintic

$$x^3y^2 + y^2z^3 + axy^2z(x + z) + (by^4 + cx^2z^2)(z - x) = 0, \tag{5}$$

and the transformations form a diedral G_8, which contains a four-group of each kind as well as the cyclic G_4.

The quintic has the geometrical properties of equation (3). The inflexional tangent $x - z = 0$ has also ordinary contact at $(1, 0, 1)$. From O there are six bitangents, and one of the points I or J is an acnode.

These six groups, composed of two G_2, three G_4, and one G_8, are the only ones without higher prime periods; so we pass now to the consideration of the transformations of period 3.

8. A cyclic G_3 can not interchange the systems of generators; for evidently, in interchange of systems, the period of the transformation must be even. There are then two kinds of G_3 to be considered: one which leaves neither system invariant, and one which leaves one system invariant. The first kind may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \omega x_2, & \omega^2 y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega^2 y, & z, & \omega w \end{pmatrix},$$

and, in the plane,

$$\begin{pmatrix} x, & y, & z \\ x, & \omega y, & z \end{pmatrix},$$

where $\omega^3 = 1$.

The substitution is made in (I) as before, and the conditions for invariance obtained. There are three resulting equations, but two are discarded because they are of lower genus. The third is

$$x_1^3y_2^3 + x_2^3y_1^3 + a(x_1^3y_1^3 + x_2^3y_2^3) + bx_1x_2^2y_1^2y_2 + cx_1^2x_2y_1y_2^2 = 0,$$

or, in space coördinates,

$$x^3 + z^3 + a(y^3 + w^3) + xz(bx + cz) = 0.$$

This projects into the sextic

$$y^3(x^3 + z^3) + a(y^6 + x^3z^3) + xy^3z(bx + cz) = 0. \tag{6'}$$

It is evident, on account of the symmetry of the equation, that the systems of generators can also interchange, and the equation possesses the following transformations of period 2:

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & y_1, & \omega x_2, & \omega^2 x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega w, & z, & \omega^2 y \end{pmatrix},$$

or, in the plane,

$$\begin{pmatrix} x, & y, & z \\ xy, & \omega xz, & yz \end{pmatrix}.$$

The centers for these space collineations are $(0, 1, 0, -\omega)$, and the planes of perspective are $y - \omega w = 0$.

Equation (6) belongs therefore to a diedral G_6. The planes $x + kz = 0$ are invariant under all transformations of the group. The centers of the three cubic cones of perspective are on the line $x = z = 0$. By the transformations of period 3 the cones are interchanged. Therefore a plane $x + kz = 0$ which is tangent to one cone is tangent to all three. Through any point on the line connecting the vertices, there are six lines tangent to a section made by a plane which contains the point but not the line, for such a section is a non-singular cubic and is therefore of class 6. There are thus six planes through BD each tangent to one cubic cone and hence tangent to all. A plane through BD tangent to the cones touches the sextic in three points, one in each plane of perspective, and at each of the eighteen such points of tangency there is a stationary plane. The twenty-seven inflexional tangent planes must each touch S_6 in but two points.

Corresponding to the above properties of S_6, the plane sextic has six tritangents from O, which accords with the class of the curve 18. Each of the points of tangency determines an adjoint conic with four-point contact. The points of tangency, like those in (1), lie by sixes on twelve adjoint conics. There are twenty-seven adjoint conics with double three-point contact.

The last S_6 does not pass through a vertex of the reference tetraedron, so that the projection from any vertex is a sextic with two triple points. A binodal quintic is obtained by inversion of the sextic, the triangle of inversion

being selected with all the vertices O, I, and J on the sextic, I and J being at the triple points. This gives the quintic

$$\begin{aligned} ay^5 - 3a^{1/3}y^4z + 3a^{2/3}y^3z^2 + x^3y^2 + (1-a^2)y^2z^3 + bx^2y^2z + cxy^2z^2 \\ - 3a^{1/3}x^3yz - 2a^{1/3}bx^2yz^2 - a^{1/3}cxyz^3 + 3a^{2/3}x^3z^2 + a^{2/3}bx^2z^3 = 0, \quad (6) \end{aligned}$$

which is transformed into itself by

$$\begin{pmatrix} x, & y, & z \\ \omega x\{\omega^2 y + a^{1/3}(1-\omega^2)z\}, & y\{\omega^2 y + a^{1/3}(1-\omega^2)z\}, & yz \end{pmatrix}$$

and by

$$\begin{pmatrix} v, & y, & z \\ vy, & vz, & yz \end{pmatrix},$$

where $v \equiv \omega x + a^{1/3}y$.

The same quintic may be obtained by transforming S_6 to a new reference tetraedron $A'BCD'$ whose vertex D' is at $(a^{1/3}, 0, 0, -1)$ on S_6 and whose reference planes are $x' = x + a^{1/3}w$ and $y' = y + a^{1/3}z$. The projection into a quintic is then made from the new vertex D'. The transformations from the original tetraedron to the plane are

$$\begin{pmatrix} x, & y, & z, & w \\ xy - a^{1/3}xz, & y^2 - a^{1/3}yz, & yz, & xz \end{pmatrix}.$$

The planes $x + kz = 0$, which as before stated are invariant under all operations of the group, go into the pencil of conics $xy - a^{1/3}xz + kyz = 0$ through OIJ. So the six points cut out on C_5 by any conic of the pencil can only interchange among themselves. Six conics of the pencil are triply tangent to C_5, since six planes of the corresponding axial pencil are triply tangent to S_6. They correspond also to the six tritangents from O in the plane sextic. Through each of the eighteen points of tangency there are likewise adjoint conics which have four-point contact with C_5.

The above transformations of period 2 are clearly inversions, with v, y, z as sides of the triangle of inversion. There are three centers O_1, O_2 and O_3, corresponding to the three values of ω in the line $v \equiv \omega x + a^{1/3}y = 0$. The coördinates of these centers referred to the OIJ triangle are $(a^{1/3}, -\omega, 0)$, corresponding to the coördinates $(a^{1/3}, -\omega, 0, 1)$ of the vertices of the cubic cones referred to the tetraedron $A'BCD'$. The three centers are points of intersection of $z = 0$ with C_5. They are not, however, inflexions; hence there are fourteen tangents to C_5 from each center besides the one at the center.

If S_6 be projected from A or C, the transformations will be all linear of the form

$$\begin{pmatrix} y, & z, & w \\ \omega z, & \omega^2 y, & w \end{pmatrix} \text{ and } \begin{pmatrix} y, & z, & w \\ \omega y, & \omega^2 z, & w \end{pmatrix};$$

and the plane sextic is

$$y^3w^3 + z^6 + az^3(y^3 + w^3) + yz^2w(byw + cz^2) = 0. \tag{6''}$$

9. Consider all groups whose operations do not interchange the systems of generators, which contain the above G_3 as an invariant subgroup. Such a group could not be a cyclic G_6, for there are not six points of S_6 to permute cyclically on an invariant generator. It can not be an octaedron group, for that contains a cyclic G_4 which is excluded on the same ground as the G_6. It can not be a diedral G_{12}, for that contains a cyclic G_6. There remain only the diedral G_6 and the tetraedral group.

It is evident that the space sextic (6′), when $b = c$, is invariant also under

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \omega x_1, & \omega^2 y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z, & \omega^2 w, & x, & \omega y \end{pmatrix},$$

or

$$\begin{pmatrix} x, & y, & z \\ yz, & \omega xz, & xy \end{pmatrix}$$

for the plane sextic projected from D.

The second sextic (6″), when $b = c$, has the quadric transformation

$$\begin{pmatrix} y, & z, & w \\ \omega^2 zw, & yw, & \omega yz \end{pmatrix}.$$

The above transformations, compounded with those for the previous diedral G_6, give transformations of periods 3 and 6, making up a diedral G_{12}, one-half of whose operations interchange the systems of generators.

The quintic (6), when $b = c$, becomes

$$\begin{aligned} ay^5 - 3a^{1/3}y^4z + 3a^{2/3}y^3z^2 + x^3y^2 + (1 - a^2)y^2z^3 + bx^2y^2z + bxy^2z^2 \\ - 3a^{1/3}x^3yz - 2a^{1/3}bx^2yz^2 - a^{1/3}bxyz^3 + 3a^{2/3}x^3z^2 + a^{2/3}bx^2z^3 = 0; \end{aligned} \tag{7}$$

and it has in addition to the transformations of the diedral G_6 the following:

$$\begin{pmatrix} x, & y, & z \\ y\{\omega^2 z + a^{1/3}(y - a^{1/3}z)\}, & x\{\omega^2 z + a^{1/3}(y - a^{1/3}z)\}, & x(y - a^{1/3}z) \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ z(y + a^{1/3}\omega^2 x), & (\omega y + a^{1/3}x)(y - a^{1/3}z), & x(y - a^{1/3}z) \end{pmatrix}.$$

10. Assuming that the subgroup which does not change the generating systems is a tetraedron group, since this group contains an invariant axial four-group, it is convenient to start with the equation of the S_6 for (4) and find what further conditions exist among the coefficients to admit the collineation

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \pm i(x_1 \pm x_2), & (x_1 \mp x_2), & (y_1 \mp y_2), & \mp i(y_1 \pm y_2) \end{pmatrix}.$$

These conditions are $a = 3$ and $b = -c$, and the resulting equation is

$$(x_1y_2 + x_2y_1)^3 + b(x_1y_1 - x_2y_2)(x_1y_1 + x_2y_2)(x_2y_1 - x_1y_2) = 0,$$

or

$$(x + z)^3 + b(y - w)(y + w)(x - z) = 0,$$

from which is obtained

$$y^3(x + z)^3 + b(y^2 - xz)(y^2 + xz)(x - z) = 0. \tag{8}$$

These equations are invariant not only under all the substitutions of the tetraedron group, but also under the remaining substitutions of the octaedron group, so that in addition to the substitutions for (4) the quintic has

$$\begin{pmatrix} x, & y, & z \\ \pm xz + xy \pm y^2 - yz, & (\pm i)^\epsilon(\pm xz + xy \mp y^2 - yz), & \mp xz + xy \mp y^2 + yz \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (x \mp iy)(y \pm iz), & (i)^\epsilon(x \pm iy)(y \pm iz), & (x \pm iy)(y \mp iz) \end{pmatrix},$$

where $\epsilon = 1, 2$.

Under all operations of the octaedron group, the plane $x + z = 0$ and its pole P or $(1, 0, 1, 0)$ with respect to F_2 are invariant. Hence the set of six bisecants from P is invariant. There are but six bisecants from P, for if there were an infinite number, there would be a central involution

$$\begin{pmatrix} x, & y, & z, & w \\ x, & -w, & z, & -y \end{pmatrix},$$

and this substitution does not leave (8) invariant. The tangents to S_6 at the points in the invariant plane are invariant as a whole under the transformations of the group. Hence they must either lie in the invariant plane or pass through the invariant point. They can not lie in the invariant plane, for in that case S_6 would have twelve points in the plane; so they must pass through P. Moreover, the six points on S_6 in the invariant plane are in six-fold involution.

11. The octaedron group G_{24} has six central and three axial involutions, and the vertices of the six cubic cones on which the space sextic lies are in the

invariant plane. They are the vertices of a complete quadrilateral whose diagonal triangle, together with the point P, determines the tetraedron whose three pairs of edges are axes of the axial involutions. The planes of perspective pass through P and cut the plane $x+z=0$ in six lines, which are the sides of a complete quadrangle whose diagonal triangle is the same as that for the complete quadrilateral. The diagonal triangle cuts F_2 in the six points on S_6. The set of planes and lines from P to the lines and points just described in the invariant plane, is invariant as a whole under the operations of the group.

Since the six planes of perspective and three axes of involution all go through P, it is evident that if P be taken as center* for the projection of S_6, a plane sextic will be obtained all of whose transformations of period 2 are linear. Moreover, as the other operations of G_{24} are generated by those of period 2, all the transformations in the group must be linear. Choose the invariant plane as plane of projection and the diagonal triangle as reference triangle. Then the equation of the F_2 section may be written

$$x^2+y^2+z^2=0,$$

and the points of S_6 in the plane are at $(0, \pm i, 1)$, $(1, 0, \pm i)$ and $(\pm i, 1, 0)$. At these points C_6 has cusps.

The above conic is invariant under six central homologies; viz.,

$$\begin{pmatrix} x, & y, & z \\ x, & z, & y \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ z, & y, & x \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ y, & x, & z \end{pmatrix},$$
$$\begin{pmatrix} x, & y, & z \\ -x, & z, & y \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ z, & -y, & x \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ y, & x, & -z \end{pmatrix};$$

and also under the following homologies, which are the projections of axial involutions in space:

$$\begin{pmatrix} x, & y, & z \\ -x, & y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ x, & y, & -z \end{pmatrix}.$$

It is evident that a sextic invariant under the above transformations is symmetric in the variables and contains no odd powers. It has the form

$$a(x^6+y^6+z^6)+b(x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4)+cx^2y^2z^2=0.$$

* The preceding equations (1)–(7) can likewise be projected from the invariant point P, and each is thus transformed into a sextic with six cusps lying on a conic.

By transferring the origin to one of the points at which there is a cusp, it is found that $b = 3a$ for genus 4.* So the equation can be written

$$(x^2 + y^2 + z^2)^3 + kx^2y^2z^2 = 0. \qquad (8')$$

This form shows that the axes are bicuspidal tangents.

The six-cuspidal sextic is of class 12, with 27 bitangents and 24 inflexions. From each center of the first six homologies there can be but six ordinary tangents to the points where the corresponding axes a_i cut C_6. Hence, from each center there are three double tangents, which means eighteen besides the cuspidal double tangents, each of which should be counted as three. This accounts for the entire number. The inflexional tangents meet in pairs on each of the six axes.

The binodal quintic (8) has, like (4), a point of inflexion at O, from which there are six bitangents. The inflexional tangent is $x - z = 0$, and it is also tangent to C_5 at the point of intersection with $y = 0$. The projection of P from D, or $(1, 0, 1)$, and the line $x + z = 0$ are invariant under all operations of the group. The six vertices of the complete quadrilateral project into points on $y + w = 0$. They are centers of quadric transformations of the group.

12. The G_3 already considered left neither system of generators invariant. Assume next that one system, say the y-system, is unchanged. Then G_3 can be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \omega x_1, & x_2, & y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \omega y, & \omega z, & w \end{pmatrix}.$$

This substitution in the general form (I) gives three equations, two of genus less than 4 and the other of the following form:

$$x_1^3 f_3(y_1, y_2) + x_2^3 \phi_3(y_1, y_2) = 0,$$

an equation containing eight constants. Of these eight, two can be absorbed in x_1 and x_2 respectively, and by a linear transformation

$$y_1 = ay_1' + by_2', \quad y_2 = cy_1' + dy_2'$$

the roots of f_3 can be prescribed. This leaves three moduli.

* For certain values of the remaining modulus more double points may appear. Thus, for $k = -27$, $p = 0$. It is projectively equivalent to the astroid or four-cusp hypocycloid.

Instead, however, of reducing the number of arbitrary constants, the symmetry will be better preserved by writing the equation in the following form:

$$x_1^3(y_1+a_1y_2)(y_1+b_1y_2)(y_1+c_1y_2)+x_2^3(y_1+a_2y_2)(y_1+b_2y_2)(y_1+c_2y_2)=0,$$

or, in space coördinates,

$$(y+a_1z)(y+b_1z)(y+c_1z)+(x+a_2w)(x+b_2w)(x+c_2w)=0.$$

Projected from D, the result is the plane sextic

$$y^3(y+a_1z)(y+b_1z)(y+c_1z)+x^3(y+a_2z)(y+b_2z)(y+c_2z)=0, \tag{9'}$$

with the transformations

$$\begin{pmatrix} x, & y, & z \\ \omega x, & y, & z \end{pmatrix}$$

into itself. The sextic has triple points at $x=y=0$ and $y=z=0$. The center of homology is at the latter point, and the axis $x=0$ crosses the curve in three points of inflexion whose tangents pass through the center. Tangents at the center can have no residual points; hence they must be inflexional. Three inflexional tangents at the triple point, together with the three from the triple point to the points on the axis, make up 18, the class of the curve. There remain thirty inflexional tangents which meet by threes on the x-axis, the thirty points of inflexion lying by threes on ten lines through the center.

Since S_6 does not pass through a vertex of the reference tetraedron, it is convenient in order to determine the quintic (9) to select a new tetraedron whose vertex D' is at $(1, 0, 0, -a_2)$. If for x and y the variables $x'-a_2w$ and $y'-a_2z$ be substituted, the equation referred to the tetraedron $ABC'D'$ is

$$\begin{aligned}\{y'+(a_1-a_2)z\}\{y'+(b_1-a_2)z\}\{y'+(c_1-a_2)z\}& \\ +x'\{x'+(b_2-a_2)w\}\{x'+(c_2-a_2)w\}&=0.\end{aligned}$$

Projected from the vertex D', this gives the quintic

$$\begin{aligned}y^2\{y+(a_1-a_2)z\}\{y+(b_1-a_2)z\}\{y+(c_1-a_2)z\}& \\ +x^3\{y+(b_2-a_2)z\}\{y+(c_2-a_2)z\}&=0,\end{aligned} \tag{9}$$

which is transformed into itself by the same transformation as the last sextic.

This quintic has a double point at I, a cusp at J, and does not pass through O. The axis $y=0$ is the cuspidal tangent. The axis $z=0$ intersects the quintic in three points of inflexion. The tangents at these points pass through I, the center of the collineation. The cuspidal tangent and three inflexional tangents from the node count as seven, leaving eight to be accounted for by the tangents at

the node. In this case the inflexional tangents at points on the z-axis can not be again tangent at the node. It is therefore necessary that the nodal tangents have four-point contact.

No two of the constants can be equal, for if a_2 is equal to b_2 or c_2 the quintic breaks up. If a_2 is equal to a_1, b_1, or c_1, there is a triple point which reduces the genus. By symmetry it is evident that no two of the coefficients can be equal.

There are twenty-four inflexional tangents and fifty-four bitangents meeting in sets of threes on the z-axis. The points of inflexion lie by threes on eight lines through the center.

13. In order that the last S_6 may go into itself by any operation which changes the y_1 and y_2, the functions f_3 and ϕ_3 must either go into themselves or interchange. In the latter case x_1 and x_2 interchange. In the first case the Hessians H_f and H_ϕ must go into themselves. Assume that the Hessians are different. Each defines a pair of points. One pair may be the basis points of a G_2 which interchanges the other pair, or both pairs may be interchanged by the G_2, in which case the basis points are given by the functional determinant of H_f and H_ϕ. Neither of these cases is possible, for in order that the three points of $f_3=0$ and those of $\phi_3=0$ be put into themselves by a G_2, one of each must be invariant and form basis points. So to obtain groups of higher orders with G_3 as subgroup, f_3 and ϕ_3 must be interchangeable, and the six factors of $f_3\phi_3$ can be arranged so as to be cyclically permuted in one or more cycles. Denoting the six factors by $a_1, a_2, a_3, b_1, b_2, b_3$, it is evident, since an a_i goes into b_j, that there is a possibility of a cycle of six elements or of three cycles of two. When, moreover, f_3 goes into ϕ_3, H_f goes into H_ϕ. The latter can be done only by collineations of period 4 or 2. Hence the collineation is of period 2.

If then the collineation is assumed to be of the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & x_1, & y_1, & -y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ y, & x, & -w, & -z \end{pmatrix},$$

the equation of the last space sextic in its first factored form is invariant when $a_1=-a_2$, $b_1=-b_2$, $c_1=-c_2$. The equation, after change of scale, becomes

$$(y+z)(y+az)(y+bz)+(x-w)(x-aw)(x-bw)=0.$$

This projects from D into the sextic

$$y^3(y+z)(y+az)(y+bz)+x^3(y-z)(y-az)(y-bz)=0, \tag{10'}$$

which belongs to the diedral G_6 whose transformations of period 3 are the same as those of the preceding equation, and one of period 2 is

$$\begin{pmatrix} x, & y, & z \\ y^2, & xy, & -xz \end{pmatrix}.$$

This sextic has of course the geometrical properties of (9′).

To obtain the plane quintic, transfer the above S_6 to a new tetraedron of reference whose vertex D' is at $(1, 0, 0, 1)$, with $x' = x - w$ and $y' = y - z$. Then

$$\{y' + 2z\}\{y' + (a-1)z\}\{y' + (b-1)z\} + x'\{x' - (a-1)w\}\{x' - (b-1)w\} = 0$$

gives the quintic

$$y^2\{y + 2z\}\{y + (a-1)z\}\{y + (b-1)z\} + x^3\{y-(a-1)z\}\{y-(b-1)z\} = 0. \quad (10)$$

The transformations of this quintic in addition to those for the last are

$$\begin{pmatrix} x, & y, & z \\ xz, & (2x-y)z, & \omega(2x-y)y \end{pmatrix} \text{ or } \begin{pmatrix} y, & z, & v \\ vz, & \omega vy, & yz \end{pmatrix},$$

where $v = 2x - y$.

The curve has the same general form as (9). There are fifteen tangents from J, since the class is 15. The cuspidal tangent counts for three, leaving twelve ordinary, six inflexional, or three inflexional and three ordinary tangents from J.

It is unnecessary to consider the forms where f_8 and ϕ_8 are identical, because the genus is lower in that case.

14. Since the four points of the Hessians can be paired in two ways, f_8 and ϕ_8 are each invariant under a transformation other than identity which is the product of two others. Such is true of

$$x_1^3y_2(ay_1^2 + y_2^2) + x_2^3y_1(y_1^2 + ay_2^2) = 0.$$

Or, putting x_i for y_i and y_i for x_i,

$$x_2y_1^3(ax_1^2 + x_2^2) + x_1y_2^3(x_1^2 + ax_2^2) = 0,$$

which projects from D into the quintic

$$xy^2(ay^2 + x^2) + z^3(y^2 + ax^2) = 0. \quad (11)$$

The collineation group of the S_6 is the diedral G_{12} which is generated by

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ -x_1, & x_2, & y_1, & -\omega y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & -y, & \omega z, & -\omega w \end{pmatrix}$$

and

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_2, & \pm x_1, & \pm\omega y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega z, & \pm\omega w, & x, & \pm y \end{pmatrix}.$$

The corresponding transformations in the plane are

$$\begin{pmatrix} x, & y, & z \\ x, & -y, & \omega z \end{pmatrix} \text{ and } \begin{pmatrix} x, & y, & z \\ yz, & \pm xz, & \omega xy \end{pmatrix}.$$

The curve passes through O, at which point $x=0$ is an inflexional tangent. The lines $x^2+ay^2=0$ are also inflexional tangents.

15. Consider next the case where H_f and H_ϕ are identical. The equation can then be put into the form

$$x_1^3y_2^3+x_1^3y_1^3+x_2^3y_2^3+a^3x_2^3y_1^3=0, \text{ or } a^3x^3+y^3+z^3+w^3=0.$$

This projects into the sextic

$$a^3x^3y^3+y^6+y^3z^3+x^3z^3=0, \tag{12'}$$

with triple points at I and J, or $x=y=0$ and $y=z=0$, and with three points of inflexion on the z-axis, tangents at which pass through the triple point as before.

The curves belong to a G_{36} generated by the following transformations:

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & x_2, & \omega y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega x, & \omega y, & z, & w \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}$$

for the plane curve; and

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \omega x_2, & y_1, & y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \omega x, & y, & z, & \omega w \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ \omega x, & y, & z \end{pmatrix}.$$

The combination of these two groups of order 3 gives a G_9, invariant in the entire group. Moreover x_1 and x_2, y_1 and y_2 can also interchange by the collineation

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ ax_2, & x_1, & y_2/a, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ z/a, & w, & ax, & y \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ yz/a, & xz, & axy \end{pmatrix}.$$

This, taken with the G_9, forms a G_{18} simply isomorphic with the transformation group of the general plane cubic into itself. This is a subgroup of the G_{36} in which the systems interchange by

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & y_1, & x_2, & x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & w, & z, & y \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ xy, & xz, & yz \end{pmatrix}.$$

The lines $x = z = 0$ and $y = w = 0$ are invariant in this group. The vertices of six cubic cones of perspective lie three on each of the invariant lines.

The lines are interchanged by the substitution

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & -x_2, & y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -w, & z, & y, & -x \end{pmatrix}, \text{ or } \begin{pmatrix} x, & y, & z \\ -xz, & yz, & y^2 \end{pmatrix}.$$

This belongs to the curve when $a^3 = -1$; that is,

$$y^3 + z^3 + w^3 - x^3 = 0,$$

or, for the plane,

$$y^6 + y^3z^3 + x^3z^3 - x^3y^3 = 0. \tag{13'}$$

This curve has thirty-six more collineations, and belongs therefore to a G_{72}.

The G_{72} is a permutation group of six things arranged in pairs of triads, the pairs being interchangeable. The G_{72} possesses three invariant sub-groups G_{36}.

The last S_6 lies on six R_5, in addition to the nine R_5 on which the preceding S_6 lies.

To obtain the quintic for the preceding equation, a change of coördinates is made as before. The new vertex D' can be taken at any one of the three points $(0, 0, 1, -t)$ on the line DC, where $t^3 = 1$. Here t is used instead of ω to distinguish between the transformations. By putting $z' = z + tw$ and $y' = y + tx$, we obtain the equation

$$(a^3 - 1)x^3 + y'^3 - 3txy'^2 + 3t^2x^2y' + z'^3 - 3tz'^2w + 3t^2z'w^2 = 0,$$

which projects into the quintic

$$(a^3 - 1)x^3y^2 + y^5 - 3txy^4 + 3t^2x^2y^3 + y^2z^3 - 3txyz^3 + 3t^2x^2z^3 = 0. \tag{12}$$

The transformation group G_{36} for the quintic (12) is generated by the following:

$$\begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ xy, & y\{\omega y + t(1-\omega)x\}, & z\{\omega y + t(1-\omega)x\} \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, \\ z(y-tx), & z(ax+ty-t^2x), & ay(ax+ty-t^2x) \end{pmatrix},$$

and

$$\begin{pmatrix} x, & y, & z \\ xy, & y(z+ty), & zy - t(xy + y^2 - xz) \end{pmatrix}.$$

As in the case of the last two quintics, there is a node which is the center of collineation of period 3, and the axis crosses the curve in points of inflexion, tangents at which pass through the center. There is likewise a cusp whose tangent passes through the node.

The G_{72} has also the following transformation for the quintic:

$$\begin{pmatrix} x, & y, & z \\ xz, & z(2tx-y), & y(2tx-y) \end{pmatrix}.$$

16. It remains to consider the case in which the curve will be invariant under a G_5. By collineations of odd periods the systems of generators are never interchanged. On the four fixed generators three points can not be permuted cyclically in fives. The three points on each generator must be at the vertices, the curve touching one generator and intersecting the other at each vertex. The five collineations can be put in the form

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \theta x_1, & \theta^4 x_2, & \theta^2 y_1, & \theta^3 y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \theta^2 y, & \theta^3 z, & \theta w \end{pmatrix},$$

where $\theta^5 = 1$. The curve has the equation

$$x_1^3 y_1^2 y_2 + x_1^2 x_2 y_2^3 + x_1 x_2^2 y_1^3 + a^5 x_2^3 y_1 y_2^2 = 0, \text{ or } x^2 y + y^2 z + z^2 w + a^5 x w^2 = 0.$$

It has, in addition to the above, the five collineations

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ a^2\theta^3 x_2, & x_1/a, & \theta y_2, & y_1/a \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ \theta z/a, & a^2\theta^4 w, & a\theta^3 x, & y/a^2 \end{pmatrix}.$$

The equation belongs therefore to the diedral G_{10}.

The corresponding quintic is

$$x^2y^3 + y^4z + xyz^3 + a^5x^3z^2 = 0, \tag{14}$$

and its transformations are

$$\begin{pmatrix} x, & y, & z \\ \theta^3 x, & y, & \theta z \end{pmatrix} \text{ and } \begin{pmatrix} x, & y, & z \\ \theta yz/a, & a^2\theta^4 xz, & a\theta^3 xy \end{pmatrix}.$$

From the equation it is seen that $z=0$ is tangent to the quintic at (0, 1, 0) and likewise at (1, 0, 0), which latter point is also a cusp. The other node is at (0, 0, 1), at which point $x=0$ is an inflexional tangent.

All transformations of the group leave the conics $a^{5/2}x^2 \pm yz = 0$ invariant. Each quadric transformation leaves invariant the four corresponding conics $a^3\theta^2 xz \pm y^2 = 0$ and $a^{1/2}\theta^{1/2}xy \pm z^2 = 0$. This makes six conics invariant under

each quadric transformation. They are projections of the partial intersections of F_3 with certain quadric cones. The straight line intersections, as in (4), are not invariant under the transformations.

From O there are either ten ordinary or five double tangents besides OI and OJ, while from I and J there are ten ordinary or five inflexional tangents.

It appears that there can be no cyclic group of higher order than 5. The G_{60} is a subgroup of the generalized icosaedron group or the symmetric G_{120} which remains to be discussed.

There are five cyclic G_4 whose generating operations may be written

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ \theta y_1, & \theta^2 y_2, & -\theta^3 x_2, & x_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -\theta w, & -x, & \theta^2 y, & \theta^3 z \end{pmatrix},$$

which becomes, for the plane,

$$\begin{pmatrix} x, & y, & z \\ \theta xz, & xy, & -\theta^2 y^2 \end{pmatrix},$$

which transformations will belong to (14) when $a^5 = -1$. The new quintic is then

$$x^2 y^3 + y^4 z + x^3 z^2 - xyz^3 = 0, \tag{15}$$

which is one form of the Bring* Curve. It possesses the linear transformations into itself whose types as given by Gordan† for the G_{60} are

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ -x_2, & x_1, & -y_2, & y_1 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -z, & w, & -x, & y \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ x_1, & \theta^3 x_2, & \theta y_1, & \theta^2 y_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ x, & \theta^2 y, & \theta^3 z, & \theta w \end{pmatrix},$$

and

$$\begin{pmatrix} x_1, & x_2, \\ \{(\theta+\theta^4)x_1 + x_2\}/(\theta^2-\theta^3), & \{x_1 - (\theta+\theta^4)x_2\}/(\theta^2-\theta^3), \\ y_1, & y_2 \\ \{(\theta^2+\theta^3)y_1 + y_2\}/(\theta^4-\theta), & \{y_1 - (\theta^2+\theta^3)y_2\}/(\theta^4-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} x, & y, \\ x + (\theta^2+\theta^3)y + z - (\theta+\theta^4)w, & (\theta^2+\theta^3)x - y + (\theta+\theta^4)z + w, \\ z, & w \\ x + (\theta+\theta^4)y + z - (\theta^2+\theta^3)w, & -(\theta+\theta^4)x + y - (\theta^2+\theta^3)z - w \end{pmatrix}.$$

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* Klein: "Vorlesungen über das Ikosaeder."

† Gordan: "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Ann.*, Vol. XIII.

These collineations, taken with

$$\begin{pmatrix} x_1, & x_2, & y_1, & y_2 \\ y_2, & -y_1, & x_1, & x_2 \end{pmatrix} = \begin{pmatrix} x, & y, & z, & w \\ -y, & z, & w, & -x \end{pmatrix},$$

generate the G_{120}. The collineations for the plane quintic are

$$\begin{pmatrix} x, & y, & z \\ yz, & -xz, & xy \end{pmatrix}, \begin{pmatrix} x, & y, & z \\ \theta x, & \theta^2 y, & z \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z \\ \{(\theta^2+\theta^3)x+y\}\{(\theta+\theta^4)y+z\}, & \{x-(\theta^2+\theta^3)y\}\{(\theta+\theta^4)y+z\}, & \{x-(\theta^2+\theta^3)y\}\{y-(\theta+\theta^4)z\} \end{pmatrix},$$

and

$$\begin{pmatrix} x, & y, & z \\ y^2, & yz, & -xz \end{pmatrix}.$$

17. The space sextic for (15) can be expressed in pentaedral coördinates as the intersection of

$$F_3 \equiv \sum_{i=1}^{5} x_i^3, \quad F_2 \equiv \sum_{i=1}^{5} x_i^2, \quad F_1 \equiv \sum_{i=1}^{5} x_i$$

in space of four dimensions. Since, however, the last equation is linear, the curve lies in three-dimensional space, and one of the coördinates can be eliminated by means of the linear equation. There are then five reference planes

$$x_1 = 0, \quad x_2 = 0, \ldots, \quad x_5 = -(x_1 + x_2 + x_3 + x_4) = 0.$$

The pentaedron of reference is thus a tetraedron plus a fifth plane which does not pass through a vertex of the tetraedron. Any plane is cut by the four others in a complete quadrilateral. Through each of the ten vertices of the pentaedron three planes pass, and on each of the ten edges there are three vertices. It is clear that the curve belongs to the symmetric G_{120}.

The collineations which permute two of the coördinate axes are central involutions. Thus

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_3, & x_5, & x_4 \end{pmatrix}$$

has the center at $(0, 0, 0, 1)$ of the reference tetraedron $x_1x_2x_3x_4$, and the plane of perspective

$$x_4 - x_5 = x_1 + x_2 + x_3 + 2x_4 = 0.$$

The plane of perspective cuts $x_4=0$ in the invariant line $x_1+x_2+x_3=0$ on which lie the three invariant vertices $(1, -1, 0, 0)$, $(1, 0, -1, 0)$ and $(0, 1, -1, 0)$. The other six vertices of the pentaedron lie in pairs on lines through the center and are interchanged by the above collineation. There are ten central perspectivities, corresponding to the ten vertices of the pentaedron. Hence the curve lies on ten cubic cones. The equation of any one of these cubic cones is obtained by eliminating one coördinate from F_2 and F_3 by means of F_1 and combining the results to eliminate a second coördinate. The result is the square of a cubic in the remaining three coördinates.

Now the symmetric G_{120} or generalized icosaedron group has five octaedron subgroups, each obtained by permuting all the variables but one. Let $x_1x_2x_3x_5$ be the set of variables. Then the quadrilateral in the x_4-plane is invariant as a whole, but its sides and vertices are interchanged by the collineations. The sides are

$$x_1=0,\ \ x_2=0,\ \ x_3=0,\ \ x_1+x_2+x_3=0.$$

The equation of the plane sextic (8′) invariant under the symmetric G_{24} or octaedron group was found to be of the form

$$(x_1^2+x_2^2+x_3^2)^3+kx_1^2x_2^2x_3^2=0.$$

The question naturally arose as to what value of k would give a sextic invariant not only under the G_{24} but also under the G_{120}. The plane sextic (8′) was referred to the diagonal triangle of the invariant quadrilateral. This suggested that the equations of S_6 be referred to the tetraedron whose base is the diagonal triangle

$$x_1+x_2=0,\ \ x_2+x_3=0,\ \ x_3+x_1=0,$$

and whose vertex is the pole $(1, 1, 1, -4)$ of $x_4=0$ with respect to F_2. Denoting the new coördinates by x, y, z, w, the transformation of coördinates may be written

$$T^{-1}\equiv\left(\frac{x,}{2x_2+2x_3+x_4,}\ \ \frac{y,}{2x_1+2x_3+x_4,}\ \ \frac{z,}{2x_1+2x_2+x_4,}\ \ \frac{w}{x_4}\right),$$

or

$$T\equiv\left(\frac{x_1,}{-x+y+z-w,}\ \ \frac{x_2,}{x-y+z-w,}\ \ \frac{x_3,}{x+y-z-w,}\ \ \frac{x_4}{4w}\right).$$

The equations of S_6 become

$$F_2'\equiv x^2+y^2+z^2+5w^2=0,$$
$$F_3'\equiv 5w^3-w(x^2+y^2+z^2)-2xyz=0.$$

The last equation combines with the preceding and reduces to $5w^3 - xyz = 0$. Eliminating w, we obtain a sextic

$$(x^2 + y^2 + z^2)^3 + 5x^2y^2z^2 = 0, \tag{15'}$$

which may be regarded as a sextic cone containing S_6, or as the plane projection of S_6 from a vertex of the new reference tetraedron upon its opposite face.

18. One of the collineations of period 5 may be written

$$R \equiv \begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_5, & x_1, & x_2, & x_3, & x_4 \end{pmatrix} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \\ -(x_1 + x_2 + x_3 + x_4), & x_1, & x_2, & x_3 \end{pmatrix}.$$

This operation transformed through T gives

$$R' \equiv T^{-1}RT \equiv \begin{pmatrix} x, & y, & z, & w \\ x+y+3z-5w, & x-3y-z-5w, & -3x+y-z-5w, & x+y-z-w \end{pmatrix}.$$

To project this operation upon the plane $w = 0$ from the opposite vertex of the reference tetraedron, we may substitute the value of w in terms of x, y, z obtained from F_2' and F_3'; that is, $w = -xyz/(x^2 + y^2 + z^2)$. Then R' for the plane is

$$\begin{pmatrix} x, & y, & z \\ (x + y + 3z)(x^2 + y^2 + z^2) - 5xyz, & (x - 3y - z)(x^2 + y^2 + z^2) - 5xyz, & (-3x + y - z)(x^2 + y^2 + z^2) - 5xyz \end{pmatrix}.$$

The basis points of the transformation are at $(0, 1, \pm i)$, $(1, 0, \pm i)$, $(1, \pm i, 0)$, which points are cusps on the sextic. To the point of intersection of two lines will correspond three points of intersection of the two corresponding cubics. Hence the transformation is birational for the curve and not for the whole plane. We have thus an example of a periodic Riemann transformation.

Others can be obtained of periods 2, 3, 4 and 5 by the same method. The following one is of period 2:

$$\begin{pmatrix} x, & y, & z \\ (3x - y - z)(x^2 + y^2 + z^2) + 5xyz, & (-x + 3y - z)(x^2 + y^2 + z^2) + 5xyz, & (-x - y + 3z)(x^2 + y^2 + z^2) + 5xyz \end{pmatrix},$$

which comes from

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_3, & x_5, & x_4 \end{pmatrix}.$$

The other nine of period 2 are linear. Thus

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ x_1, & x_2, & x_5, & x_4, & x_3 \end{pmatrix} \text{ corresponds to } \begin{pmatrix} x, & y, & z \\ y, & x, & -z \end{pmatrix}.$$

Twenty-three operations of the G_{120} are linear, and the remaining ninety-six are Riemann transformations differing from those given above only in the signs, and the position of the coefficient 3.

§ 2. *The Conical Case.*

19. When the space sextic lies upon a cone K_2 of the second order, the projection from a point A on the surface will be either a tacnodal quintic or a sextic with three branches which have a common tangent, according as A lies upon or without the curve.

Let the tetraedron of reference be $ABCD$, with DA and DB generators of K_2, and DAB the polar plane of DC with respect to K_2. Then the equation of K_2 is

$$xy - z^2 = 0.$$

When K_2 is projected from A, the transformation may be written

$$\begin{pmatrix} x, & y, & z, & w \\ x'^2, & y'^2, & x'y', & y'z' \end{pmatrix}.$$

The plane section

$$ax + by + cz + dw = 0$$

is thus transformed into the conic

$$ax'^2 + by'^2 + cx'y' + dy'z' = 0,$$

which is tangent to $y' = 0$ at $x' = 0$. The vertices OIJ of the reference triangle are at the intersections of AB, AD, and AC, with the plane of projection. For simplicity in writing, the accents will be dropped from the plane coördinates.

The pencil of planes through AD defines the system of generators of K_2. They project into a pencil of lines $x = ky$, each line of which cuts the curve in three points besides those at I, thus defining a point-group series g_3^1.

All other sections of K_2 through A have as images lines which do not contain I. When A is not on S_6, these lines cut the plane curve C_6 in six variable points, and define a g_6^2. When A is on S_6, its image is at A', the intersection of the tangent at A with the line IJ.

The space sextic S_6 is defined by the quadric cone K_2 and a cubic surface F_3. The equation of the cubic surface may be written

$$w^3 + w^2 f_1(x, y, z) + w f_2(x, y, z) + f_3(x, y, z) = 0.$$

This is easily transformed to a cubic in which there is no second term. It is therefore written

$$w^3+w(a_0x^2+a_1xz+a_2z^2+a_3yz+a_4y^2+a_5xy)+b_0x^3+b_1x^2z+b_2xz^2$$
$$+b_3z^3+b_4yz^2+b_5y^2z+b_6y^3+b_7x^2y+b_8xy^2+b_9xyz=0.$$

By combining the above with the equation $xy=z^2$, four terms, viz., a_5xyw, b_7x^2y, b_8xy^2 and b_9xyz, can be united with other terms. The equation is thus reduced to the form

$$F_3\equiv w^3+w(a_0x^2+a_1xz+a_2z^2+a_3yz+a_4y^2)+b_0x^3+b_1x^2z+b_2xz^2$$
$$+b_3z^3+b_4yz^2+b_5y^2z+b_6y^3=0,$$

which, taken with K_2, defines S_6.

The above projects into

$$C_6\equiv y^3z^3+yz\sum_{i=0}^{4}a_ix^{4-i}y^i+\sum_{i=0}^{6}b_ix^{6-i}y^i=0,$$

an equation of a plane sextic with three branches tangent to the y-axis at I or $(0, 0, 1)$. When, however, $b_0=0$, the factor y can be removed and the quintic will have a tacnode at I, tangent likewise to the y-axis.

The problem now is to find all collineations which leave K_2 and F_3 invariant and to derive the corresponding plane transformations.

20. The following central involutions leave K_2 invariant:

$$E^2\equiv\begin{pmatrix}x, & y, & z, & w\\ y, & x, & z, & w\end{pmatrix},\quad F^2\equiv\begin{pmatrix}x, & y, & z, & w\\ -y, & -x, & z, & w\end{pmatrix},\quad G^2\equiv\begin{pmatrix}x, & y, & z, & w\\ x, & y, & -z, & w\end{pmatrix},$$

and a fourth,

$$\begin{pmatrix}x, & y, & z, & w\\ x, & y, & z, & -w\end{pmatrix}.$$

The last is discarded, because it can not leave F_3 invariant, except when $f_3(xy)$ vanishes identically, which happens only when the genus is less than four. These transformations project from A into

$$\begin{pmatrix}x, & y, & z\\ xy, & x^2, & yz\end{pmatrix},\quad \begin{pmatrix}x, & y, & z\\ xy, & -x^2, & yz\end{pmatrix},\quad \begin{pmatrix}x, & y, & z\\ -x, & y, & z\end{pmatrix}$$

respectively.

Three axial involutions are generated by the four central involutions, viz.,

$$E^2F^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & -y, & z, & w \end{pmatrix}, \quad F^2G^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ y, & x, & z, & -w \end{pmatrix}, \quad E^2G^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ y, & x, & -z, & w \end{pmatrix},$$

which project into

$$\begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ xy, & x^2, & -yz \end{pmatrix}, \quad \begin{pmatrix} x, & y, & z \\ -xy, & x^2, & yz \end{pmatrix}.$$

The reason for using the exponent 2 with E, F, and G will appear later.

The cubic surface F_3 also is invariant under G^2 when

$$a_1 = a_3 = b_1 = b_3 = b_5 = 0,$$

and the equation is

$$w^3 + w(a_0x^2 + a_2z^2 + a_4y^2) + b_0x^3 + b_2xz^2 + b_4yz^2 + b_6y^3 = 0, \qquad (1)$$

or, in the plane,

$$y^3z^3 + yz(a_0x^4 + a_2x^2y^2 + a_4y^4) + b_0x^6 + b_2x^4y^2 + b_4x^2y^4 + b_6y^6 = 0. \qquad (1')$$

The coefficients b_2 and b_4 can be absorbed by change of scale.

From the center C or $(0, 0, 1, 0)$ there are six tangents to (1), and their points of contact are on the x- and y-generators in the invariant plane $z = 0$. These project into tangents from J to the plane sextic (1′) at I and at the three other points in which the curve intersects the x-axis. There are also six bitangents from J, which make up 18, the class of the curve.

When, in F_3, $a_1 = a_3 = b_0 = b_2 = b_4 = b_6 = 0$, the equation is

$$w^3 + w(a_0x^2 + a_2z^2 + a_4y^2) + b_1x^2z + b_3z^3 + b_5y^2z = 0, \qquad (2)$$

an equation invariant under the axial homology E^2F^2. It projects into the quintic

$$y^2z^3 + z(a_0x^4 + a_2x^2y^2 + a_4y^4) + b_1x^5 + b_3x^3y^2 + b_5xy^4 = 0. \qquad (2')$$

The plane quintic (2′) has an inflexional tangent at O, an ordinary tangent from O to the point of intersection with the y-axis, and six bitangents from O.

21. To obtain a four-group of the first kind, with two central and one axial involution, one may find the condition that E^2 shall leave either (1) or (2) invariant. The former yields a sextic; so the latter is chosen as being the simpler. The equation then is

$$w^3 + w\{a_0(x^2 + y^2) + a_2z^2\} + b_1z(x^2 + y^2) + b_3z^3 = 0. \qquad (3)$$

The plane curve is

$$y^2z^3 + z\{a_0(x^4 + y^4) + a_2x^2y^2\} + b_1x(x^4 + y^4) + b_3x^3y^2 = 0. \qquad (3')$$

Either b_1 or b_3 can be divided out and absorbed in w.

The vertices V_1 and V_2 of the two cubic cones of perspective for (3) are on AB and project into O. The curve (3′) passes through O, and the tangent $a_0z + b_1x = 0$ is an inflexional tangent at O and an ordinary tangent on the y-axis. This tangent is the image of a double osculating plane $a_0w + b_1z = 0$, which is an inflexional tangent plane to both cubic cones.

An equation invariant under a four-group of the second kind may be obtained from (2) by finding the conditions that it be invariant under F^2G^2. The equation thus reduces to

$$w^3 + w\{a_0(x^2 + y^2) + a_2z^2\} + b_1z(x^2 - y^2) = 0. \qquad (4)$$

This gives

$$y^2z^3 + z\{a_0(x^4 + y^4) + a_2x^2y^2\} + b_1x(x^4 - y^4) = 0. \qquad (4')$$

The last coefficient reduces to unity.

This quintic has an inflexional tangent $a_0z - b_1x = 0$ which is not tangent a second time to the curve.

From (2) is also obtained an equation invariant under a diedral group of order 8. The generating operations of a cyclic G_4 which leaves K_2 invariant may be written

$$EF \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & y, & -iz, & iw \end{pmatrix},$$

which projects into

$$\begin{pmatrix} x, & y, & z \\ x, & iy, & -z \end{pmatrix}.$$

Equation (2), if invariant under EF, reduces to

$$w^3 + a_2wz^2 + z(b_1x^2 + b_5y^2) = 0, \qquad (5)$$

which is manifestly invariant also under E^2 and F^2. This projects into the quintic

$$y^2z^3 + a_2x^2y^2z + x(b_1x^4 + b_5y^4) = 0. \qquad (5')$$

The last two coefficients can be changed to unity.

To each of the five preceding groups there is a corresponding group in § 1.

22. There is also a cyclic group of order 4, whose collineation of period 2 is the central involution G^2. Its generating operation is

$$G \equiv \begin{pmatrix} x, & y, & z, & w \\ -x, & y, & iz, & w \end{pmatrix},$$

which projects into

$$\begin{pmatrix} x, & y, & z \\ ix, & y, & z \end{pmatrix}.$$

This differs essentially from the last group, which contains an axial involution. The equations belonging to this group are

$$w^3 + w(a_0x^2 + a_4y^2) + b_2xz^2 + b_6y^3 = 0 \tag{6}$$

and

$$y^2z^2 + z(a_0x^4 + a_4y^4) + b_2x^4y + b_6y^5 = 0. \tag{6'}$$

The coefficients a_0 and a_4 can be absorbed. There is no corresponding equation in § 1.

The center of the homology is at J. The tangent $a_0z + b_2y = 0$ has contact of the fourth order. This counts as five tangents and JI counts as two. There are, moreover, from J, three tangents to points on the x-axis, each with contact of the third order.

23. The transformations

$$H_y \equiv \begin{pmatrix} x, & y, & z, & w \\ \omega^2x, & \omega y, & z, & w \end{pmatrix},$$

where $\omega^3 = 1$, form a cyclic G_3. Here every point of the polar line $x = y = 0$ is invariant. The plane transformations are of the form

$$\begin{pmatrix} x, & y, & z \\ x, & \omega y, & z \end{pmatrix}.$$

The subscript y distinguishes this transformation from one following, where the axis of homology is the x-axis. The equations are

$$w^3 + a_2z^2w + b_0x^3 + b_1y^3 + b_2z^3 = 0 \tag{7}$$

and

$$y^3z^3 + a_2x^2y^3z + b_0x^6 + b_1y^6 + b_2x^3y^3 = 0. \tag{7'}$$

The three central involutions

$$K \equiv \begin{pmatrix} x, & y, & z, & w \\ b_1^{1/3}\omega y, & \omega^2 b_0^{1/3}x, & b_0^{1/6}b_1^{1/6}z, & b_0^{1/6}b_1^{1/6}w \end{pmatrix},$$

respectively

$$\begin{pmatrix} x, & y, & z \\ b_0^{1/6}b_1^{1/6}xy, & \omega b_0^{1/3}x^2, & b_0^{1/6}b_1^{1/6}yz \end{pmatrix},$$

leave the curves invariant. Hence they belong to a diedral G_6. The three centers of perspective V_1, V_2, V_3 lie on AB and project into O. By a change of scale the coefficients of x^3 and y^3 can be reduced to unity. This simplifies the last transformations. The planes of perspective then become $x-\omega y=0$, which project into the lines $x^2-\omega y^2=0$.

This curve has six tritangents from O, as there are six planes through AB tangent to all three cones of perspective. The sextic (6′), § 1, corresponds to this one.

When, in (7), $b_3=0$, the central involution G^2 leaves the curve invariant and the group is a diedral group of order 12. The equations are

$$w^3+a_2z^2w+x^3+y^3=0 \tag{8}$$

and

$$y^3z^3+a_2x^2y^2z+x^6+y^6=0. \tag{8'}$$

24. There is also a cyclic G_3 which leaves invariant all points of the x-generator. Its transformations are

$$H_x\equiv\begin{pmatrix} x, & y, & z, & w \\ \omega^2x, & \omega y, & z, & \omega w\end{pmatrix},$$

which project into

$$\begin{pmatrix} x, & y, & z \\ \omega x, & y, & z\end{pmatrix}.$$

The equations are

$$w^3+w(a_1xz+a_4y^2)+b_0x^3+b_6y^3+b_9z^3=0 \tag{9}$$

and

$$y^3z^3+y^2z(a_1x^3+a_4y^3)+b_0x^6+b_6y^6+b_3x^3y^3=0. \tag{9'}$$

The coefficients b_0 and b_6 can be absorbed by change of scale.

Attention has been called to differences in the space transformations H_x and H_y. The corresponding plane homologies differ merely in respect to the curve; the one has a secant through the singular point as axis of homology, while the other has the tangent at that point as axis.

If, in equation (9′), the function $f_6(x,y)$ be the sextic covariant of $f_4(x,y)$, the equations, after absorbing coefficients, may be written

$$w^3+aw(xz+y^2)+x^3+20z^3-8y^3=0 \tag{10}$$

and

$$y^3z^3+ay^2z(x^3+y^3)+x^6+20x^3y^3-8y^6=0. \tag{10'}$$

The space and plane sextics are left invariant by the four-group whose respective transformations are the following:

$$\begin{pmatrix} x, & y, & z, & w \\ x+4y-4z, & x+y+2z, & -x+2y+z, & -3w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (-x+2y)(x+y), & (x+y)^2, & -3yz \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, & w \\ \omega^2 x+4y-4\omega z, & \omega x+\omega^2 y+2z, & -x+2\omega y+\omega^2 z, & -3\omega^2 w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (-\omega x+2y)(\omega^2 x+y), & (\omega^2 x+\omega y)^2, & -3\omega^2 yz \end{pmatrix},$$

$$\begin{pmatrix} x, & y, & z, & w \\ \omega x+4y-4\omega^2 z, & x+\omega^2 y+2\omega z, & -\omega^2 x+2\omega y+z, & -3\omega^2 w \end{pmatrix}$$

and

$$\begin{pmatrix} x, & y, & z \\ (2y-\omega^2 x)(\omega y+x), & (\omega y+x)^2, & -3\omega^2 yz \end{pmatrix}.$$

The group is therefore the tetraedron group generated by the above four-group and the preceding group of order 3.

A cyclic G_6 can be obtained from the transformations H_x and G^2. Their product gives

$$\begin{pmatrix} x, & y, & z, & w \\ \omega^2 x, & \omega y, & -z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ -\omega x, & y, & z \end{pmatrix};$$

and the equations are

$$w^3 + a_4 y^2 w + b_0 x^3 + b_6 y^3 = 0 \tag{11}$$

and

$$y^3 z^3 + a_4 y^5 z + b_0 x^6 + b_6 y^6 = 0, \tag{11'}$$

where b_0 and b_6 can be absorbed. All points of the x-generator are invariant.

A cyclic G_{12} is obtained whose transformations may be represented by

$$G^3E^2F^2H_x \equiv \begin{pmatrix} x, & y, & z, & w \\ -\omega^2 x, & y, & i\omega z, & -w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \omega x, & -iy, & iz \end{pmatrix}.$$

The equations for this group are

$$w^3 + a_4 y^2 w + b_0 x^3 = 0 \tag{12}$$

and

$$y^3 z^3 + a_4 y^5 z + b_0 x^6 = 0; \tag{12'}$$

and both coefficients can be absorbed by change of scale.

25. For the cyclic G_5 we may have

$$L \equiv \begin{pmatrix} x, & y, & z, & w \\ \theta^3 x, & y, & \theta z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^5 = 1$; and the equations are

$$w^3 + a_4 y^2 w + b_1 x^2 z + b_6 y^3 = 0 \tag{13}$$

and

$$y^2 z^3 + a_4 y^4 z + b_1 x^5 + b_6 y^5 = 0. \tag{13'}$$

The coefficient b_1 can be dropped.

This is a subgroup of the cyclic G_{10} whose operations are

$$LE^2F^2 \equiv \begin{pmatrix} x, & y, & z, & w \\ -\theta^3 x, & -y, & \theta z, & w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ \theta x, & -y, & z \end{pmatrix},$$

with the equations

$$w^3 + a_4 y^2 w + b_1 x^2 z = 0, \tag{14}$$

$$y^2 z^3 + a_4 y^4 z + b_1 x^5 = 0. \tag{14'}$$

Here both coefficients can be changed to unity.

26. In the following cases $f_4(x, y)$ vanishes identically. There will always be a central collineation of period 3, viz.,

$$H_z \equiv H_x H_y \equiv \begin{pmatrix} x, & y, & z, & w \\ x, & y, & z, & \omega w \end{pmatrix} \text{ or } \begin{pmatrix} x, & y, & z \\ x, & y, & \omega z \end{pmatrix}.$$

The equations are

$$w^3 + b_0 x^3 + b_1 x^2 z + b_2 x z^2 + b_3 z^3 + b_4 y z^2 + b_5 y^2 z + b_6 y^3 = 0 \tag{15}$$

and

$$y^3 z^3 + \sum_{i=0}^{6} b_i x^{6-i} y^i = 0. \tag{15'}$$

The following equations, with their transformations, arise from special values of the coefficients b_i:

$$w^3 + b_0 x^3 + b_2 x z^2 + b_4 y z^2 + b_6 y^3 = 0, \tag{16}$$

$$y^3 z^3 + b_0 x^6 + b_2 x^4 y^2 + b_4 x^2 y^4 + b_6 y^6 = 0, \tag{16'}$$

whose transformations are G^2 respectively G'^2, which with H_z respectively H'_z form a cyclic G_6. The accents as before are used to denote the plane transformations. The coefficients b_0 and b_6 can be absorbed.

The curve

$$w^3 + b_1 x^2 z + b_3 z^3 + b_5 y^2 z = 0 \tag{17}$$

has the axial involution E^2F^2. It projects into

$$y^3 z^3 + b_1 x^5 + b_3 x^3 y^2 + b_5 x y^4 = 0. \tag{17'}$$

Now as b_1 and b_2 can be absorbed, the transformation E^2 belongs to the group. The group is of order 12, with a four-group and three cyclic groups of order 6 as subgroups.

If, in (16), $b_2 = b_4$ and $b_0 = b_6$, we have

$$w^3 + x^3 + y^3 + b_2 z^2(x + y) = 0, \tag{16a}$$

with the transformations E, G and K. This sextic belongs to a G_{12} of the same kind as that for (17). It projects into a sextic.

There is also

$$w^3 + b_0 x^3 + b_3 z^3 + b_6 y^3 = 0, \tag{18}$$

with the transformations H_x, H_y and H_z. It projects into

$$y^3 z^3 + b_0 x^6 + b_3 x^3 y^3 + b_6 y^6 = 0. \tag{18'}$$

Since b_0 and b_6 are reducible to unity, there is also the central perspectivity E^2. Thus a group of order 18 is generated.

When $b_3 = 0$, the equation

$$w^3 + b(x^3 + y^3) = 0 \tag{19}$$

is invariant under the perspectivity G^2. It projects into

$$y^3 z^3 + b(x^6 + y^6) = 0. \tag{19'}$$

The group is a G_{36} with the above G_{18} as invariant subgroup. We can absorb b.

When, in (17), $b_3 = 0$, we have

$$w^3 + z(x^2 + y^2) = 0, \tag{20}$$

$$y^2 z^3 + x(x^4 + y^4) = 0, \tag{20'}$$

with the transformations EF, respectively $E'F'$. These equations belong to a G_{72} with the G_{18} as invariant subgroup.

Last of all there are the equations

$$w^3 + b_1 x^2 z + b_6 y^3 = 0, \tag{21}$$

$$y^2 z^3 + b_1 x^5 + b_6 y^5 = 0, \tag{21'}$$

where b_1 and b_6 can be omitted. These curves belong to a cyclic G_{15} generated by H_z, L and H_z', L' respectively.

27. We have seen that space sextics of genus 4 may be projected into plane sextics with six distinct double points if the center of projection be selected without the hyperboloid or cone. The double points moreover will lie on a

conic.* Thus if three of the double points are collinear, the remaining three must be either collinear or coincident.

Any C_6 with six distinct double points can be transformed by aid of adjoint cubics into a quintic with two nodes either distinct or coincident; thence, by projecting from the corresponding space sextic, into a C_6 whose six double points lie on a conic.

The plane sextic with two distinct triple points may be inverted into a binodal quintic if the triangle of inversion be chosen with center O at an ordinary point of the curve and the vertices I and J at the triple points. The equation of the quintic then is

$$x^3\phi_2(y,z) + x^2\psi_3(y,z) + xyf_3(y,z) + y^2\phi_3(y,z) = 0,$$

with nodes at $(1, 0, 0)$ and $(0, 0, 1)$.

This may be transformed to a new reference triangle with vertex O' at a node, and I' and J' ordinary points of the quintic collinear with the second node. By inversion a sextic is obtained with two double points and a triple point having two of its branches tangent. The triple point thus counts as four double points. The transformation from the original sextic is of order 3.

The binodal quintic may likewise be inverted into a sextic with a triple point and three non-collinear double points by selecting the triangle of reference with O' at a node and I' and J' at ordinary points on the quintic not collinear with a node. Two of these double points unite in a tacnode when one of the vertices I' or J' is at the point where the quintic intersects the line which joins the nodes.

The plane sextic with two coincident triple points may be transformed into a quintic with a tacnode by using the quadric transformation with OIJ coincident at the tacnode and a base conic tangent to the tacnodal tangent. The arbitrary constant of the quadric transformation is so chosen that the transformed curve will be a quintic.

Finally, if the triangle of inversion be selected with O at the tacnode and I and J ordinary points of the curve, such that neither OI nor OJ will be the tacnodal tangent, the inverted curve is the sextic† with a triple point at O and three double points on IJ.

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* Halphen: Mémoire sur la Classification des Courbes gauches algébriques, *Journal de l'École Polytechnique*, cahier LII (1882), pp. 1–200.

† L. Kraus: Note über aussergewöhnliche Specialgruppen auf algebraischen Curven, *Math. Ann.*, Vol. XVI (1879).

A sextic with a quadruple point possesses a g_2^1, is therefore hyperelliptic and belongs to the following section.

§ 3. *The Hyperelliptic Curves.*

28. As was noted at the beginning of this paper, the hyperelliptic C_m (4) can not be transformed to a C_5. For when C_m is hyperelliptic, $m \overline{\gtrless} p+2$, or, in this case, $m \overline{\gtrless} 6$. Reducing the equation to the canonical form $y^2 = f_{2p+2}(x)$ gives $y^2 = f_{10}(x)$, or, in homogeneous coördinates,

$$y^2z^8 = \sum_{n=0}^{10} a_n x^{10-n} z^n, \tag{1}$$

which admits the transformation

$$T \equiv \begin{pmatrix} x, & y, & z \\ x, & -y, & z \end{pmatrix}.$$

If the coefficients of the odd powers of x all vanish, the equation

$$y^2z^8 = a_0x^{10} + a_2x^8z^2 + a_4x^6z^4 + a_6x^4z^6 + a_8x^2z^8 + a_{10}z^{10} \tag{2}$$

belongs to the four-group whose operations are

$$\begin{pmatrix} x, & y, & z \\ \pm x, & \pm y, & z \end{pmatrix}.$$

When $a_0 = a_{10}$, $a_2 = a_8$ and $a_4 = a_6$, the curve

$$z^8y^2 = a_0(x^{10} + z^{10}) + a_2(x^8z^2 + x^2z^8) + a_4(x^6z^4 + x^4z^6) \tag{3}$$

possesses, in addition to the above four-group, the transformation

$$R \equiv \begin{pmatrix} x, & y, & z \\ x^4z, & yz^4, & x^5 \end{pmatrix},$$

an operation of the second order commutative with the other operations, and the equation belongs therefore to a diedral G_8.

When, in (3), $a_2 = a_4 = 0$, the equation

$$y^2z^8 = a_0(x^{10} + z^{10}) \tag{4}$$

possesses, in addition to the above transformation of G_8, a cyclic G_{10} whose operations are

$$S \equiv \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^{10} = 1$, and the group of the equation is of order 40. For consider the generating operations R, S and T. There are ten operations of the form SR,

including R itself. There are nine more of the form RS, but they are the same as those of the form SRT and must not be counted twice. There are ten of the form S, including identity. These twenty operations, combined with T, form the G_{40}.

Returning to equation (1), it may be that $a_n = a_{10-n}$ for $n = 1, 2, \ldots, 10$. The equation is then of the form

$$y^2z^8 = a_0(x^{10}+z^{10}) + a_1xz(x^8+z^8) + a_2x^2z^2(x^6+z^6) + a_3x^3z^3(x^4+z^4) \\ + a_4x^4z^4(x^2+z^2) + a_5x^5z^5, \qquad (5)$$

which is found to possess the transformations T and R of the above group, but not S. This equation, like (2), belongs to a four-group, and it reduces to (3) with the diedral G_8 when $a_1 = a_3 = 0$.

If $a_1 = a_2 = a_3 = a_4$ in equation (5), we have

$$y^2z^8 = a_0(x^{10}+z^{10}) + a_5x^5z^5, \qquad (6)$$

which possesses the subgroup, a cyclic G_5 with generating operations

$$Q \equiv \begin{pmatrix} x, & y, & z \\ \theta x, & y, & z \end{pmatrix},$$

where $\theta^5 = 1$. This equation has therefore five transformations Q_iR, including R itself, five of form Q, and combining with T a group G_{20} is found, a subgroup of the G_{40} in equation (4). While R and Q are not commutative, it is clear that $Q_iR = RQ_j$.

If the equation had the form of (6), except that the coefficients of x^{10} and z^{10} are unequal, then

$$y^2z^8 = a_0x^{10} + a_5x^5z^5 + a_{10}z^{10} \qquad (7)$$

has the transformations Q and T, but not R. Hence the group is a cyclic G_{10}.

If the curve be reduced to the form

$$y^2 = x^{2p+1} + 1 \text{ or } y^2z^7 = x^9 + z^9, \qquad (8)$$

the curve has

$$\begin{pmatrix} x, & y, & z \\ mx, & \pm y, & z \end{pmatrix},$$

When the equation is of the form

$$y^2 = x(x^{2p} + 1) \text{ or } y^2z^7 = x(x^8 + z^8), \tag{9}$$

it has the transformation R, and also

$$V \equiv \begin{pmatrix} x, & y, & z \\ n^2x, & ny, & z \end{pmatrix},$$

where $n^{16} = 1$. The operation V is of period 16. The operations V and R are not commutative, but any product RV is one of the products VR, and the square of either is

$$\begin{pmatrix} x, & y, & z \\ x, & n^8y, & z \end{pmatrix} \equiv V^8,$$

which is identity or T, according to which one of the sixteenth roots of unity n is. The operations are V^i and RV^i, where $i = 0, 1, \ldots, 15$. The group is a G_{32}, neither diedral nor cyclic.

In closing, let me state that in certain cases details have been given which might have been omitted had Wiman's paper been more easily accessible to the reader.

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On Some Loci Associated with Plane Curves.

BY DR. C. H. SISAM.

1. Let there be given two algebraic curves C and C_1, lying in the same plane. Let t (Fig. 1) be a tangent to C_1. At the points of intersection of t and C construct the tangents to C. Then as t describes the system of tangents to C_1, the points of intersection of the tangents to C describe a locus, some of the properties of which will now be determined. This locus will be referred to as the locus C_1'. The given curves C and C_1 will be supposed to have only the ordinary point and line singularities and to lie in general position with respect to each other.

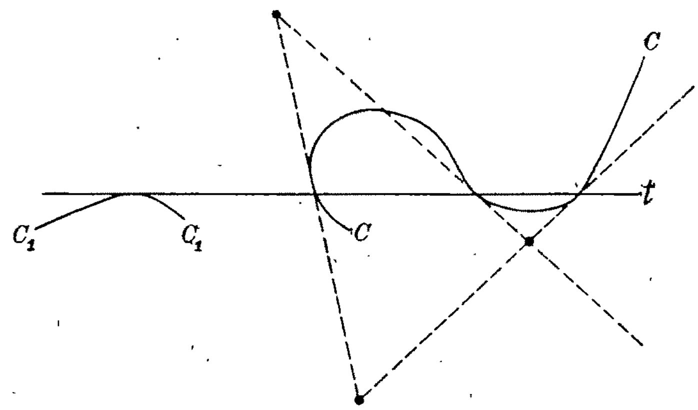

FIG. 1.

2. To determine the order of C_1', consider first the particular case in which C_1' reduces to the pencil of lines through a point P. The locus so obtained is a direct generalization, to curves of order higher than the second, of the polar of a point with respect to a conic, according to a frequent definition of that locus. Let it be referred to as the locus C_P'.

3. The equation of C_P involves not only the current coordinates (x, y, z) of a point on the locus, but also the coordinates (x', y', z') of P. Its order in

(x', y', z') is* $\frac{1}{2}n(n-1)$, since it is the equation of the cords of contact of the tangents from (x, y, z) to C. To determine its order in (x, y, z), let $x'=\rho x$, $y'=\rho y$, $z'=\rho z$. We then obtain the equation of the locus of the points each of which lies upon its own C'_P locus. To this locus belongs each double and inflectional tangent of C. Every other point of the locus must lie upon C. But C is an $(n-1)$-fold component of the locus, for, if P lies upon C, the tangent at P is an $(n-1)$-fold component of C_P. Hence

$$\tfrac{1}{2}n(n-1)+m'_P=\tau+\iota+m(n-1).$$

Solving for m'_P and simplifying by means of Plucker's equations, we obtain:†

$$m'_P=(m-1)(n-1)-p.$$

This expression is unchanged by duality. If, in fact, we denote by C''_l the dual of C'_P, it follows immediately from the definition that $m'_P=n''_l$.

4. Returning now to the locus C'_1 (paragraph 1), we have:

$$m'_1=n_1[(m-1)(n-1)-p].$$

For, each intersection of C'_1 with an arbitrary line l is determined by means of a tangent to C_1 which touches the locus C''_l corresponding to l, and conversely. The number of intersections of C'_1 and l is, therefore,

$$m'_1=n_1n''_l=n_1[(m-1)(n-1)-p].$$

5. The locus C'_1 passes through the point of tangency to C of each tangent common to C and C_1; and at each of the remaining intersections with C of these common tangents, C'_1 has a point of inflection, the inflectional tangent coinciding with the line tangent to C at the point of intersection.

6. For, let $x=0$ be the common tangent, $(0, 0)$ the point of tangency to C, and $(0, 1)$ the point of tangency to C_1, the coordinates being non-homogeneous. Let the equation of the tangent to C_1 in the neighborhood of $x=0$ be:

$$x+ty-t+\alpha t^2+\beta t^3+\ldots=0.$$

* The following notation will be used: Let m be the order of C; n, its class; δ, the number of its double points; κ, of its cusps; τ, of its double tangents; ι, of its inflections; and p, its genus. To distinguish the corresponding numbers for another curve, as C'_P, subscripts and indices will be affixed; *e. g.*, m'_P is the order of C'_P.

† The locus C'_P was mentioned by Steiner (*Gesamelte Werke*, Vol. II, p. 599, No. 5). He points out that its order equals the class of the dual locus, and states, without proof, that the class of the dual locus is $\frac{1}{2}m(m-1)(2m-3)$ (*ib.*, *id.*, p. 589). This result is correct, provided that C has no multiple points. Sporer, in an article in the *Mathematisch-Naturwissenschaftliche Mitteilungen von Dr. O. Boklen*, Vol. III, pp. 55–58, 1890, attempted to prove Steiner's formula. His proof, however, is incorrect. For example, his proof is based largely on an auxiliary locus of class m^2 which is affirmed to have $m(m-1)$ double tangents, all passing through one fixed point.

Let the equations of C in the neighborhood of the origin be:

$$x = s^2, \qquad y = as + bs^2 + cs^3 + \ldots,$$

where $a \neq 0$. The equation of the tangent to C at the point whose parameter is s is, then:

$$2ys = x(a + 2bs \ldots) + as^2 - cs^4 \ldots.$$

In order that the point of C whose parameter is s lie on the tangent to C_1 whose parameter is t, we must have:

$$s^2 + t(as + bs^2 \ldots) - t + \alpha t^2 \ldots = 0.$$

The relation between s and t in the neighborhood of $s = t = 0$ is, therefore:

$$t = t'^2, \qquad s = t' - \frac{a}{2} t'^2 + \frac{a^2 - 4b - 4\alpha}{8} t'^3 \ldots.$$

To the values $\pm t'$ corresponds the same value of t, but different values of s. They therefore determine two points of C lying on the same tangent to C_1. Substituting the two values of s so obtained in the equation of the tangent to C and determining the intersection, we obtain:

$$x = -t + \left(\frac{a^2}{4} + \frac{4c}{a} + b + \alpha\right) t^2 \ldots,$$

$$y = -\frac{2b + a^2}{2} t + \ldots.$$

This is the equation of a branch of C_1' passing through the origin. It should also be noticed, for the proof of a later theorem (see paragraphs 10 and 15), that, since the coefficients of the first power of t do not vanish, the origin is not a cusp of C_1'.

7. In order to show that, at each of the remaining intersections of C with $x = 0$, C_1' has an inflection, let $(0, a')$, where $1 \neq a' \neq 0$, be another intersection of C with $x = 0$. Let the equations of C in the neighborhood of $(0, a')$ be:

$$x = r, \qquad y = a' + b'r + c'r^2 \ldots.$$

From the condition that the point whose parameter is r lie upon the tangent to C_1 whose parameter is $t = t'^2$, we obtain:

$$r = (1 - a') t'^2 + (a'b' - b' - \alpha) t'^4 \ldots.$$

Substituting in the equation of the tangent, and solving as simultaneous with the equation of the tangent at the point whose parameter is s, we obtain:

$$x = \frac{2a'}{a} t' + \left[\frac{4a'}{a^2}(b' - b) - a' - 1\right] t'^2 \ldots,$$

$$y = a' + \frac{2a'}{a} b' t' + b' \left[\frac{4a'}{a^2}(b' - b) - a' - 1\right] t'^2 \ldots.$$

The tangent to this branch of C_1' at $(0, a')$ is $y = a' + b'x$, which coincides with the tangent to C at that point. If, in this equation of the tangent, we substitute the above expressions for x and y, the coefficients of the powers of t', of degree less than three, vanish. Hence, $y = a' + b'x$ is an inflectional tangent to C_1'. Since a generic line through $(0, a')$ meets the curve in only one point, $(0, a')$ is not a cusp (see paragraph 10).

8. The tangents to C_1 in the neighborhood of a tangent passing through a node determine, by their intersections with C, a branch of C_1' having a cusp at the node of C. Hence, at each node of C, C_1' has n_1 cusps. For, let $(0, 0)$ be a node of C and $x = 0$ be a tangent to C_1. Let the equations of C in the neighborhood of $(0, 0)$ be:

$$x = s, \qquad y = as + bs^2 + cs^3 \dots ;$$
$$x = r, \qquad y = a'r + b'r^2 + c'r^3 \dots .$$

The tangents to the two branches being distinct, $a \neq a'$. Let the equation of the tangent to C_1 be as in paragraph 6. Imposing the condition that the points s and r lie on the tangent t, we obtain for the locus of the points of intersection of corresponding tangents to C:

$$x = \frac{b-b'}{a-a'}t^2 + 2\frac{(a-a')(c-c') - (a-a')(ab-a'b') - \alpha(a-a')(b-b') + (b-b')^2}{(a-a')^2}t^3 \dots ,$$

$$y = \frac{a'b - ab'}{a - a'}t^2 \dots .$$

Hence C_1' has a cusp at the origin.

9. Through each cusp of C pass n_1 simple branches each touching the cuspidal tangent. For, let the equations of C in the neighborhood of the origin be:

$$x = s^2, \qquad y = as^3 + bs^4 + \dots ,$$

the equation of the tangent to C_1 in the neighborhood of $x = 0$ being as in paragraph 6. Then the locus of the points of intersection of the tangents to C at the points which lie on the same tangent to C_1 is found to be:

$$x = \frac{1}{3}t + \frac{4c - 3a\alpha}{9a}t^2 + \dots ,$$

$$y = -\frac{1}{3}bt^2 \dots .$$

These are the equations of a branch of C_1' having the origin for a simple point and touching the cuspidal tangent, $y = 0$, at that point.

10. We have already determined (paragraph 8) δn_1 of the cusps of C_1'.

The total number of cusps of C_1' will now be determined. For this purpose, it will be convenient to distinguish several cases, a number of which have been discussed in the four paragraphs immediately preceding.

11. Consider next, therefore, the form of C_1' in the neighborhood of the point of intersection of a cuspidal tangent to C and the tangent at another intersection with C of a tangent to C_1 through the cusp. In addition to the data of paragraph 9, let $x=0$ meet C at $(0, a')$ and let the form of C in the neighborhood of $(0, a')$ be:

$$x = r, \qquad y = a' + b' r + c' r^2 \ldots\ldots$$

We may suppose, it being projectively no restriction, that $b' \neq 0$. The equations of C_1' are then of the form:

$$x = -\frac{a'}{b'} - \frac{3\,a\,a'}{2\,b'^2}\,t + \ldots,$$

$$y = -\frac{3\,a\,a'}{b'}\,t + \ldots..$$

Hence C_1' does not have a cusp at the point of intersection of the cuspidal tangent with the tangent at $(0, a')$.

12. Consider, next, two branches of C whose equations are:

$$x = s, \qquad y = a + b s + c s^2 + \ldots;$$
$$x = r, \qquad y = a' + b' s + c' s^2 + \ldots,$$

where $a = a'$ and $b \neq b'$, the equation of the tangent to C_1 being as in paragraph 6. It will now be determined under what conditions the tangents at $(0, a)$ and $(0, a')$ determine a cusp of C_1'.

13. The locus of the point of intersection of corresponding tangents is:

$$x = -\frac{a-a'}{b-b'} + 2\,[c\,(1-a) - c'\,(1-a')]\,\frac{a-a'}{(b-b')^2}\,t + \ldots,$$

$$y = \frac{a'\,b - a\,b'}{b-b'} + 2\,[c\,b'\,(1-a) - b\,c'\,(1-a')]\,\frac{a-a'}{(b-b')^2}\,t + \ldots..$$

If this branch of C_1' has a cusp at $t=0$, the coefficients of the first power of t must vanish. But $a \neq a'$ and $b \neq b'$; hence the conditions for a cusp reduce to

$$c\,(1-a) = 0, \qquad c'\,(1-a') = 0.$$

But $c=0$ or $c'=0$ is the condition that one of the two branches of C have a point of inflection on $x=0$, and $a=1$ or $a'=1$ is the condition that one of the two branches pass through the point of tangency $(0, 1)$ of $x=0$ to C_1. Since C and C_1 are in general position with respect to each other, the above two equations can not be satisfied simultaneously.

14. There still remains to be considered the case in which $x=0$ is an inflectional tangent to C_1. Let the equation of the tangent to C_1, in the neighborhood of the inflectional tangent $x=0$, be:

$$x + t^2 y - t^2 + \beta t^3 + \ldots\ldots = 0.$$

Let the equations of C, in the neighborhood of two of its intersections with $x=0$, be:

$$x = s, \quad y = a + c s^2 + \ldots\ldots;$$
$$x = r, \quad y = r + c' r^2 + \ldots\ldots,$$

where $0 \neq a \neq 1$. Then the locus of the point of intersection of corresponding tangents is:

$$x = a + 2a(c + c' - ac)t^2 - 2\beta a(c - c')t^3 \ldots\ldots,$$
$$y = a + 2ac(1-a)t^2 - 2\beta a c t^3 \ldots\ldots.$$

Hence, C_1' has a cusp at the point of intersection (a, a) of the tangents to C at $(0, 0)$ and $(0, a)$. Hence, each point of intersection of the tangents to C at the m intersections with C of each inflectional tangent to C_1 is a cusp of C_1'.

15. Hence, in general, the number of cusps of C_1' is:

$$\varkappa_1' = n_1 \delta + \tfrac{1}{2} m(m-1)\iota_1.$$

16. The genus, p_1', of C_1' will next be determined. For this purpose, the following auxiliary locus will be used: Let q be an arbitrary line in the plane, and Q an arbitrary point on it. Draw the n_1 tangents from Q to C_1. These n_1 tangents to C_1 determine $\frac{1}{2} n_1 m(m-1)$ points of C_1'. Draw the lines joining these $\frac{1}{2} n_1 m(m-1)$ points to Q. Then the auxiliary locus in question is the locus of these joining lines as Q describes q. This line locus is in $(1, 1)$ correspondence with C_1'. Hence its genus is equal to that of C_1'.

17. The line q is an m_1'-fold line of this locus. For, each intersection of C_1' with q determines a line of the locus coincident with q. Through an arbitrary point of q pass $\frac{1}{2} n_1 m(m-1)$ other lines of the locus. Hence the class of the envelope of this system of lines is $m_1' + \frac{1}{2} n_1 m(m-1)$.

18. At each intersection, Q, other than the points of tangency, of q with the envelope of these lines or with an inflectional tangent of the envelope, two of the $\frac{1}{2} n_1 m(m-1)$ lines through Q become consecutive, and conversely. But if two of these lines are consecutive, the corresponding points of C_1' must be consecutive. This may happen in either of two ways. First, points determined by different tangents to C_1 from Q may become consecutive. The two tangents to C_1 which determine them must then be consecutive. There are $m_1 + \iota_1$ points Q at which two of the tangents to C_1 become consecutive. At each of

The tangent at the point whose parameter is t meets C again in the neighbo hood of the cusp and also in the neighborhood of $(0, a')$. The locus of the poi of intersection of the tangents at these two points is a branch of C' who equations are:

$$x = -\frac{3a'}{4a}t + \frac{9a'(b'-b)}{16a^3}t^2 + \dots,$$

$$y = a' - \frac{3a'}{4a}b't + \dots.$$

Hence C' touches C at $(0, a')$. It should also be noticed that this point is n a cusp of C'.

27. At each intersection with C, distinct from the inflection, of ea inflectional tangent, C' has a ramphoid cusp. For, let the form of C in t neighborhood of the point of inflection be:

$$x = t^3, \qquad y = at + bt^2 + ct^3 + \dots,$$

and in the neighborhood of $(0, a')$, be:

$$x = s, \qquad y = a' + b's^2 + \dots.$$

The tangent at the point whose parameter is t determines by its intersectio with C in the neighborhood of $(0, 0)$ and $(0, a')$ a branch of C' who equations are:

$$x = \frac{12a'}{a^2}t^2 + \frac{12ba' - 16a^3}{a^3}t^3 + \dots,$$

$$y = a' + \frac{63a'^2b'}{a^2}t^4 + \dots.$$

Hence the approximate form of C' in the neighborhood of $(0, a')$ is

$$y - a' - Mx^2 = Nx^4,$$

where M and N are constants. Hence C' has a ramphoid cusp at $(0, a')$, t cuspidal tangent being the tangent $y = a'$ to C at that point.

28. We have, therefore, in general, for the number of cusps of C', t expression

$$\varkappa' = \delta(n-4) + \tfrac{1}{2}\iota(m-2)(m-3).$$

29. The genus of C' may be determined in a manner analogous to that b which C_1' was obtained (paragraph 16). On an arbitrary line q take a point Draw the n tangents from Q to C, thus determining $\frac{1}{2}n(m-2)(m-3)$ poin of C'. Join these points to Q. As Q describes q, these joining lines envelo a curve of class $m' + \frac{1}{2}n(m-2)(m-3)$. The sum of the order and of t number of inflections of the envelope is

Hence the genus of this envelope, that is, the genus of C', is:

$$p' = \tfrac{1}{2}(m + \iota - 2n)(m - 2)(m - 3) + \tfrac{1}{2}[2\tau + \varkappa(n - 3)](m - 4) + 1$$

30. With the assistance of the locus C' or of its dual, it is possil determine the number of tangents t to an arbitrary algebraic curve C such the tangents to C at two of the intersections of t with C meet at a po which lies on C.* Every such point T is evidently a point of intersection

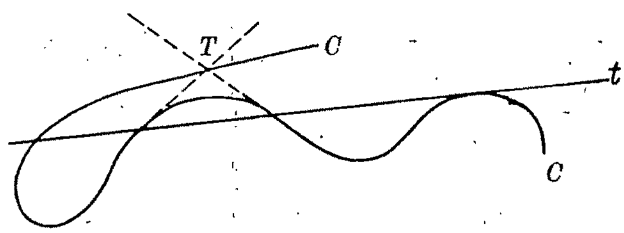

FIG. 2.

and C', and every such line t is tangent both to C and the dual of C'. number of points of intersection of C and C' is:

$$m\{n[(m-1)(n-1) - p] - 4\tau - 3\iota - m(2n - 3)\}.$$

From this number should be subtracted the number of intersections due t singularities of C (paragraphs 25–27). The resulting number of points T tangents t is:

$$mn[(m-1)(n-1) - p] - m^2(2n - 3) - 8m\tau - 7m\iota$$
$$- 4n\delta + 16\delta - 3n\varkappa + 15\varkappa + 14\tau - 2m\varkappa + 12\iota$$

31. The number in question can also be obtained by subtracting fron total number of common tangents to C and the dual of C' the number of coi tangents due to the singularities of C. We are thus led to the expression:

$$mn[(m-1)(n-1) - p] - n^2(2m - 3) - 8\delta n - 7n\varkappa$$
$$- 4m\tau + 16\tau - 3m\iota + 15\iota + 14\delta - 2n\iota + 12\varkappa$$

This expression is equivalent to the preceding one.

* On certain algebraic curves, every point has the property in question. In this case C itself is ponent of C' and of its dual. Weil has shown, in the *Nouvelles Annales de Mathématiques*, Ser. 3, V pp. 93–95, that, in the case of the curves $y^m = x^p$, if any tangent, t, to the curve be drawn and, at th of intersection of t with the curve, the tangents to the curve be constructed, then all the points of inte of these tangents lie on the given curve.

Plane Sections of a Weddle Surface.

By F. Morley and J. R. Conner.

§ 1. *An Equation of the Surface.*

The Jacobian of all quadrics on six given points of space is a Weddle quartic surface W; that is to say, W is the locus of points x, y apolar to all quadrics on the 6 points.

We use the symbol :: for "is or are apolar to." The 6 points determine a norm-curve N, which may be given by

$$x_0 = 1,\ x_1 = -t,\ x_2 = t^2,\ x_3 = -t^3. \tag{1}$$

The points are then defined by a sextic equation

$$\alpha \equiv (\alpha t)^6 = 0.$$

The quadrics on the 6 points are a 3-fold system, built on three containing the points of N, namely,

$$x_0x_2 = x_1^2,\ x_0x_3 = x_1x_2,\ x_1x_3 = x_2^2,$$

and one other not containing the points of N. That other is chosen which :: all quadrics on the planes of N. We speak of this as a quadric (of points) *orthic* to N. It is uniquely defined when the sextic α is given and may be written $(\alpha x)^2$.

The quadrics which :: all quadrics on the 6 points are then quadrics (of planes) which are orthic to N and also :: $(\alpha x)^2$.

Now if we have a quadric of points $(\alpha x)^2$ and a quadric of planes $(a\xi)^2$, both orthic to N, then it is to be shown that the binary sextic defining the points common to α and N :: the sextic defining the planes common to a and N. This we shall prove analytically. If $(a\xi)^2$ be orthic, then

$$a_{02} = a_{11},\ a_{03} = a_{12},\ a_{13} = a_{22}. \tag{2}$$

The binary sextics are

$$(a_0t^3 + 3a_1t^2 + 3a_2t + a_3)^2,\ (\alpha_0 - \alpha_1t + \alpha_2t^2 - \alpha_3t^3)^2.$$

Bearing (2) in mind, the apolarity-condition of these sextics is

$$a_{00}\alpha_{00} + 2a_{01}\alpha_{01} + a_{11}(\alpha_1^2 + 2\alpha_{02}) + 2a_{12}(\alpha_{12} + \alpha_{03}) \\ + a_{22}(\alpha_2^2 + 2\alpha_{13}) + 2a_{23}\alpha_{23} + a_{33}\alpha_{33} = 0,$$

or

$$(a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)^2 = 0.$$

That is, the sextics are apolar if the quadrics are, and conversely.*

We have now a convenient definition of a Weddle surface, when the sextic is given on N. It is the locus of points x, y representing† binary cubics $(xt)^3$, $(yt)^3$, subject to the conditions, first, that they are an orthic quadric of N, *i. e.*, that

$$\left.\begin{aligned} x_0y_2 + x_2y_0 &= 2x_1y_1, \\ x_0y_3 + x_3y_0 &= x_1y_2 + x_2y_1, \\ x_1y_3 + x_3y_1 &= 2x_2y_2, \end{aligned}\right\} \tag{3}$$

each cubic being now the Jacobian of the other; second, that $(xt)^3(yt)^3 :: (\alpha t)^6$, *i. e.*, that

$$|\alpha x|^3 |\alpha y|^3 = 0. \tag{4}$$

This equation (4) we regard as the equation of W. If from (3) and (4) we eliminate y, we have the usual form for W (Baker: Multiply Periodic Functions, p. 65).

§ 2. *Configurations on the Section.*

Written out, (4) is

$$\alpha_6x_0y_0 - 3\alpha_5(x_0y_1 + x_1y_0) + 15\alpha_4x_1y_1 - 10\alpha_3(x_0y_3 + x_3y_0) \\ + 15\alpha_2x_2y_2 - 3\alpha_1(x_2y_3 + x_3y_2) + \alpha_0x_3y_3 = 0.$$

Now if in particular the 6 points coincide by twos, $(\alpha t)^6$ may be taken as

$$\lambda t^6 + 2\lambda t^3 + \lambda = 0,$$

and (4) becomes

$$\lambda(x_0y_0 - x_0y_3 - x_3y_0 + x_3y_3) = 0,$$

and this special Weddle surface breaks up into the plane $x_0 = x_3$ which joins the 3 points, and the cubic surface into which this plane is transformed by (3).‡

* It is only required that *one* of the quadrics be orthic. The theorem seems to be a case of Study's general theory, *Math. Annalen*, Vol. XL.

† We write here the coördinates of a point x as x_{000}, x_{001}, x_{011}, x_{111}.

‡ The cubic will be briefly characterized later (§ 6).

Since (4) is linear in the coefficients a, a pencil of sextics gives a pencil of Weddle surfaces, and thus in particular the pencil of sextics

$$(at)^6 + \lambda(t^3 + 1)^2 = 0$$

gives *a pencil of surfaces* W_λ *having a common plane section* Q.

The 6 original points, $1, \ldots, 6$, are nodes of W_0; their joins, $12, \ldots$, are lines of W_0; and the 10 lines $123|456$ are also lines of W_0.

These lines meet the section Q in 15 points $12, 13, \ldots$, lying by threes on 20 lines $123, \ldots$, and in 10 points $123|456$, the whole forming a configuration, let us say Γ.*

The curve Q contains then a configuration Γ, but being common to all the surfaces W_λ it contains infinitely many such configurations. This is the *quod erat demonstrandum;* namely, that *the plane section of a Weddle surface contains infinitely many configurations* Γ. We speak of these as a pencil of configurations.

§ 3. *The Invariant Condition.*

The existence of a configuration on a curve is apt to imply that an invariant of the curve vanishes. In the present case, the equation of Q, referred to the points e_i where it meets N as a reference triangle, can be written at once. The curve is on the reference points; it cuts each reference line again in points apolar with the other reference lines, because the Weddle surface is invariant under the transformation (3); and also the tangents at the reference points are on a point, from the known fact that N is an asymptotic curve on W. Thus, using Salmon's notation for quartic curves and their invariants, Q is

$$Q \equiv a_2x^3y + a_3x^3z + b_3y^3z + b_1y^3x + c_1z^3x + c_2z^3y + 3xyz(lx + my + nz) = 0,$$

where

$$a_2b_3c_1 + a_3b_1c_2 = 0.$$

Hence the invariants A and B are

$$A = -12lmn + 12(lb_1c_1 + mc_2a_2 + na_3b_3),$$

$$B = -\begin{vmatrix} l & a_3 & a_2 \\ b_3 & m & b_1 \\ c_2 & c_1 & n \end{vmatrix}^2;$$

and therefore the invariant which vanishes is

$$A^2 + 144B.$$

* The presumption is that a quartic curve on the 15 point of Γ is unique and must therefore contain also the 10 adjoined points. This, however, has not been proved.

The vanishing of this invariant is characteristic for the plane section of a Weddle surface.

§ 4. *The Locus of Lines of the Configurations.*

We shall give a second proof of the existence of configurations on the curve, and thereby establish the locus of the lines of a pencil of configurations.

Quadrics on the 6 points meet any plane π in conics apolar to a range, Φ. Quadrics on 5 of the points, say 1, 2, 3, 4, 5, meet π in conics apolar to a definite conic C_6 — the conic defining the polarity of the Desargues configuration determined on π by 1, 2, 3, 4, 5. The six conics C_i are in the range Φ.* N carries a double infinity of quadrics on the 6 points, and hence the three points p, q, r, where N meets π, carry a double infinity of conics apolar to Φ; pqr is then the diagonal 3-point of Φ.

A quadric of the system on the 6 points may be made to touch π at any given point. The two lines, ξ, η, cut out of this quadric by $\pi :: \Phi$. In particular the two generators cut by π out of a quadric cone with vertex on $\pi :: \Phi$. There is one further condition on this pair of lines: it is that *they touch a conic of the range, say* Φ', *on any four lines of* Γ *like* 123, 124, 134, 234. For the cone determines a tetrahedral complex with 1234 as fundamental tetrahedron, and the six lines touch the complex conic on π.

We state this as a

Lemma: If a tedrahedron is inscribed in a quadric cone, any plane on the vertex meets the cone and the planes of the tetrahedron in 6 lines on a conic.

The ranges Φ and Φ' are not independent; a conic of Φ' touches the diagonal lines qr, rp, pq of Φ. For N is projected from p by a quadric cone on 1234 with vertex p, and by the lemma pq and pr touch a conic of Φ'. Similarly, rp and rq touch a conic of Φ', necessarily the same conic.

Conversely, two ranges, Φ and Φ', connected in this way, determine uniquely a configuration Γ. For choose any tetrahedron with faces on the base lines of Φ'. It easily follows from the lemma that the four vertices of this tetrahedron are on a norm-curve N with p, q, r. The point 56, into which the conic of Φ' touching qr, rp, pq is transformed by the line-transformation $\xi\eta :: \Phi$, carries a unique line bisecant to N, thus determining the points 5 and 6 on N, and thereby a configuration Γ with the associated range Φ.

* Carver, *Trans. Am. Math. Soc.*, Vol. VI, No. 4, p. 548.

We now find the locus of pairs of lines touching conics of Φ' and apolar to Φ. Choose, as the range Φ,

$$m_0x_0^2 + m_1x_1^2 + m_2x_2^2 = 0,$$
$$m_0 + m_1 + m_2 = 0;$$

pqr is then the reference triangle, and lines ξ, η apolar to Φ are connected by the relations

$$\xi_i = 1/\eta_i \qquad (i = 0, 1, 2).$$

If Φ' is $(\alpha\xi)^2 + \lambda(\beta\xi)^2 = 0$, the equation of our locus is at once

$$\Sigma \equiv \begin{vmatrix} (\alpha\xi)^2 & (\beta\xi)^2 \\ (\alpha/\xi)^2 & (\beta/\xi)^2 \end{vmatrix} = 0,$$

where $(\alpha/\xi) \equiv \alpha_0/\xi_0 + \alpha_1/\xi_1 + \alpha_2/\xi_2$. Using coördinates ξ, η, ζ for convenience of writing, and supposing Φ' to be

$$\lambda(a\eta\zeta + b\zeta\xi + c\xi\eta) + l\xi^2 + m\eta^2 + n\zeta^2 + f\eta\zeta + g\zeta\xi + h\xi\eta = 0, \tag{5}$$

we have

$$\begin{aligned}\Sigma \equiv\, & al\eta\zeta(\eta^2\zeta^2 - \xi^4) + bm\zeta\xi(\zeta^2\xi^2 - \eta^4) + cn\xi\eta(\xi^2\eta^2 - \zeta^4) \\ & + [a(m-n) + bh - cg]\xi^2\eta\zeta(\zeta^2 - \eta^2) \\ & + [b(n-l) + cf - ah]\eta^2\zeta\xi(\xi^2 - \zeta^2) \\ & + [c(l-m) + ag - bf]\zeta^2\xi\eta(\eta^2 - \xi^2) = 0.\end{aligned} \tag{6}$$

We shall show that if Σ is given in the form

$$\begin{aligned}\Sigma \equiv\, & A\eta\zeta(\eta^2\zeta^2 - \xi^4) + B\zeta\xi(\zeta^2\xi^2 - \eta^4) + C\xi\eta(\xi^2\eta^2 - \zeta^4) \\ & + F\xi^2\eta\zeta(\zeta^2 - \xi^2) + G\eta^2\zeta\xi(\xi^2 - \zeta^2) + H\zeta^2\xi\eta(\eta^2 - \xi^2) = 0,\end{aligned} \tag{7}$$

the pencil Φ' is not unique, but may be chosen in ∞^1 ways. We first find the equation of the section of W which is the locus of joins of corresponding lines of Σ. Let (α, β, γ) be a line of Σ. The join of this line and its correspondent is

$$x = \alpha(\gamma^2 - \beta^2), \quad y = \beta(\alpha^2 - \gamma^2), \quad z = \gamma(\beta^2 - \alpha^2).$$

Substituting α, β, γ in (7) and dividing by $\alpha\beta\gamma$, we have

$$\begin{aligned}& \frac{A(\beta^2\gamma^2 - \alpha^4)}{\alpha} + \frac{B(\gamma^2\alpha^2 - \beta^4)}{\beta} + \frac{C(\alpha^2\beta^2 - \gamma^4)}{\gamma} \\ & \qquad + F\alpha(\gamma^2 - \beta^2) + G\beta(\alpha^2 - \gamma^2) + H\gamma(\beta^2 - \alpha^2) = 0.\end{aligned} \tag{8}$$

Now

$$\frac{\beta^2\gamma^2 - \alpha^4}{\alpha} = \frac{y^2 - z^2}{x}, \quad \frac{\gamma^2\alpha^2 - \beta^4}{\beta} = \frac{z^2 - x^2}{y}, \quad \frac{\alpha^2\beta^2 - \gamma^4}{\gamma} = \frac{x^2 - y^2}{z}.$$

Hence (8) becomes

$$Q \equiv Ayz(y^2 - z^2) + Bzx(z^2 - x^2) + Cxy(x^2 - y^2) + xyz(Fx + Gy + Hz) = 0. \tag{9}$$

Given an equation of the form (7), we have six equations to determine $a, b, c, l, m, n, f, g, h$:

$$al = \rho A, \quad bm = \rho B, \quad cn = \rho C, \tag{10a}$$

$$\left.\begin{aligned} a(m - n) + bh - cg &= \rho F, \\ b(n - l) + cf - ah &= \rho G, \\ c(l - m) + ag - bf &= \rho H. \end{aligned}\right\} \tag{10b}$$

Multiply the first of equations (10b) by a, the second by b, the third by c, and add; then substitute the values of l, m, n from (10a). We have

$$Abc(b^2 - c^2) + Bca(c^2 - a^2) + Cab(a^2 - b^2) + abc(Fa + Gb + Hc) = 0.$$

This shows that the conic $a\eta\zeta + b\zeta\xi + c\xi\eta = 0$ must be the transform by Φ of a point of Q. If a, b, c are chosen to satisfy (9), we may choose arbitrarily any one of the quantities f, g, h, and the other two are determined in such a manner as to satisfy all the equations (10). The arbitrariness in the choice of f, g, h may be accounted for by the fact that (6) is unaltered if we put

$$f = f' + \lambda a, \quad g = g' + \lambda g, \quad h = h' + \lambda c,$$

which means merely that (5) is any conic of Φ'.

Σ may then be generated in ∞^1 ways; a point (a, b, c) of Q determines uniquely a range Φ', and thereby a configuration Γ circumscribed to Σ and inscribed to Q.

The form of equation (7) shows that not only are the reference lines double lines of Σ, but the reference points are double points, the tangents at the double point being the two reference lines.

It is worthy of note that the range Φ is the same for all configurations of the pencil.

From the form of equation (9) we infer that any triangle inscribed in N meets W in the six vertices of a 4-line. In other words, *the join of points of W on two sides of the triangle meets W on the third side.* Given W, a configuration inscribed in Q may be determined as follows:

Choose any point 1 of N. From 1 N is projected by a quadric cone which meets Q in p, q, r, and in five other points, determining on N five points 2, 3, 4, 5, 6, and on Q five points 12, 13, 14, 15, 16. It follows from the above remark that the configuration determined by the points 1, 2, 3, 4, 5, 6 is inscribed in Q.

To construct W, when a configuration Γ is given, we may take any line

on 12, and on it any two points, as 1 and 2. The points 3, 4, 5, 6 are then determined, and thereby W. For a given Γ there are then ∞^4 W's; and therefore for a given Q there are, as there should be, ∞^5 W's.

§ 5. *The Number of Pencils of Configurations.*

There will be, on the quartic curve Q, not one but several pencils of configurations. The number of pencils has not been found, nor has the preliminary question, in how many ways a quartic can be written in the form (9), that is, without even powers, been answered. But it is worth while to establish a special case, that for the Clebsch quartic, where $B = 0$, the latter number is 8. For when the equation of a quartic is thrown into the form (9), B becomes a perfect square, say $-\Delta^2$. The bordered catalecticant is

$$\begin{vmatrix} 0 & 0 & 0 & & & & \xi^2 \\ 0 & 0 & 0 & & \Delta & & \eta^2 \\ 0 & 0 & 0 & & & & \zeta^2 \\ & & & 0 & n & m & \eta\zeta \\ & \Delta & & n & 0 & l & \zeta\xi \\ & & & m & l & 0 & \xi\eta \\ \xi^2 & \eta^2 & \zeta^2 & \eta\zeta & \zeta\xi & \xi\eta & 0 \end{vmatrix},$$

and when developed, every term except the even ones has Δ as a factor. Hence, when $\Delta = 0$, the $C_{5,4}$ consists of even terms. But the $C_{5,4}$ is then the square of the apolar conic.

Hence, for a Clebsch quartic each such triangle is apolar to the conic, and each such reference line is on the reciprocal of the quartic as to the conic. But the lines already touch a $C_{8,0,6}$, the locus of lines cut harmonically. There are then 24 such lines or 8 such triangles. Any two give 6 points on a conic; we have then 28 conics which should be connected with the double lines of the curve.

If our curve Q have $B = 0$, that is, if both A and B are 0, it follows that, when it is referred to any one of the 8 triangles,

$$a_2b_3c_1 + a_3b_1c_2 = 0.$$

The quartic can then be reduced in 8 ways to the form (9), and for each such form there is a definite curve Σ and a pencil of configurations (§ 4). Thus, for a curve Q with $B = 0$ there are 8 pencils of configurations Γ. This is probably true when $B \neq 0$.

§ 6. *The Weddle Cubic.*

The degenerate case when W is a plane and a cubic is of interest. The cubic is the transform of a plane by the transformation (3).

Now this transformation sends a line into a norm-curve N'. But if the line is on a point p of N, the line p of N is part of the transform, and if the line is also on the plane p of N, the line p counts twice,* and the rest of the transform is a line. Since (3) is involutory, this line is also on a point and corresponding plane of N. Let us call such a line, which transforms into a line without being a fixed line, a *permanent* line.

A plane π passes into a cubic surface Ω with nodes at the points p_i on π and N.

In the plane π is a curve R^4 of class 3, order 4, the section of the developable of N. Curves of order 3 on π, on the cusps of R and also on the cusp-tangents, touch the planes p_i of N at the 3 cusps, and therefore their transforms are also plane cubics, sections of Ω. In this case, then, the usual mapping of a plane on a cubic surface by a 3-fold system of cubics is merely the transformation (3).† The cusp-lines, being permanent lines, in the plane π and therefore on its null-point p, are transformed into permanent lines also on a plane π' and its null-point p', p' being the transform of p. These are lines of Ω.

Thus Ω is a cubic surface with 3 nodes, and 3 lines *in a pencil.*

Any cubic surface with a norm-curve as asymptotic curve is of this kind. For the developable occurs twice in its transform, leaving a plane.

We may notice that the transforms of sections of Ω by tangent planes at points of N are *osculants* of the R^4, for, by a theorem of Mr. Thomsen, these osculants have their nodes on the R^4.

The condition that a cubic surface have 3 lines in a pencil is thevanishing of Salmon's invariant I_{100}. For if 3 lines are in a pencil, the polar quadric of the vertex p is two planes, so that p is a double point of the Hessian. But if the point $x_3 = x_4 = x_5$ is on the surface $(\varkappa x^3) = 0$, then $\varkappa_1 = \varkappa_2$; that is, $I_{100} = 0$.

BALTIMORE, *January*, 1909.

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*Schoute, *Niew Archief*, 1899, p. 97.

† The general mapping by plane cubics on 6 points is effected by the transformation xy :: net of quadrics. Conner, *Johns Hopkins Circulars*, July, 1908, p. 95.

The Differential Equations Satisfied by Abelian Theta Functions of Genus Three.

BY J. EDMUND WRIGHT.

In several papers* and in his book "Multiply Periodic Functions,"† Baker has given the differential equations satisfied by hyperelliptic theta functions. His method is most satisfactory in its final outcome, because the constants that occur in the equations are expressed in terms of the associated Riemann Surface, with definitely known cross-cuts, but owing to this complete determination of the equations the process involves long and complicated algebraic manipulation. Its application to any but the hyperelliptic functions would seem almost impossible. Now if we start from the general definition of a theta function as a uniform integral function of several variables, that possesses certain period properties, we can discover enough about its nature to enable us to give the general forms of the differential equations it satisfies, and then it appears that conditions of coexistence of these equations are sufficient to make them precise.

For example, the general nature of the theta functions of genus 2 leads us to the conclusion that it must satisfy five differential equations of the fourth order, of a certain particular form. These equations involve twenty constants, but conditions to be satisfied in order that they may coexist reduces this number, so that finally the equations depend on only three essential constants, and therefore we conclude that the general solution of the final differential equations must be such a theta function. It is possible that, the equations once obtained, they may be integrated directly, and thus the theory may be made complete from this point of view.‡

The purpose of this paper is to determine the differential equations whose general solution is a general theta function of genus 3. As a first illustration,

* *Proc. Camb. Phil. Soc.*, Vol. IX (1898), p. 517; *ibid.*, Vol. XII (1903), p. 219; *Acta Math.*, t. XXVII (1903), p. 135.

† Cambridge University Press (1907).

‡ Cf. "*Multiply Periodic Functions*," p. 44 sqq.

we propose to apply the method outlined to the case of $p=2$ to obtain Baker's results.

The case of $p=3$ leads to a division of the types of equation into two classes. The first of these turns out to be the hyperelliptic case. By adding a suitable exponential factor to the theta function the equations are given by means of covariants of certain ternary forms; these forms are: 1) a quadratic whose coefficients are the second derivatives of the logarithm of the theta function; 2) a cubic whose coefficients are the third derivatives; 3) a quartic whose coefficients are the fourth derivatives, and similarly for higher derivatives, and 4) certain fixed forms. For the hyperelliptic case the fixed forms are a conic and a quartic. These two curves determine a binary octavic, namely that cut out on the conic by the quartic, and this case is thus associated with the invariant and covariant properties of a binary octavic. In the non-hyperelliptic case there is only one fixed form, a general quartic. It thus appears that this case is closely connected with the geometrical properties of a general quartic. This is interesting in view of the fact that a non-hyperelliptic curve of genus three can always be birationally transformed into a non-singular quartic, whereas this is not true of a hyperelliptic curve of genus three, for which the reduced curve of lowest order is a quintic with a triple point.

§ 1.

In the subsequent work we need some general definitions and theorems, which we quote from Baker's "Abelian Functions." *

Suppose that we have four matrices $\omega, \omega', \eta, \eta'$, each of p rows and columns, which satisfy the conditions: 1) that the determinant of ω is not zero; 2) that the matrix $\omega^{-1}\omega' (\equiv \tau)$ is symmetrical; 3) that for real values of $n_1, n_2, \ldots, n_p$ the quadratic form $\omega^{-1}\omega' n^2$ has its imaginary part positive; 4) that the matrix $\eta\omega^{-1}$ is symmetrical; 5) that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\bar{\omega}^{-1}$. We put

$$a = \tfrac{1}{2}\eta\omega^{-1}, \quad h = \tfrac{1}{2}\pi i\omega^{-1}, \quad b = \pi i\omega^{-1}\omega',$$

so that

$$\eta = 2a\omega, \quad \eta' = 2a\omega' - \bar{h}, \quad h\omega = \tfrac{1}{2}\pi i, \quad h\omega' = \tfrac{1}{2}b;$$

and we write

$$\lambda_m(u) = H_m(u + \tfrac{1}{2}\Omega_m) - \pi i m m',$$

where

$$H_m = 2\eta m + 2\eta' m', \quad \Omega_m = 2\omega m + 2\omega' m'.$$

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* Hereafter quoted as A. F.

Also let Q, Q' denote two assigned rows of p rational quantities, and suppose $\Pi(u)$ to be an integral function of the p arguments $u_1, u_2, \ldots, u_p$ that satisfies the equation

$$\Pi(u+\Omega_m) = e^{r\lambda_m(u)+2\pi i(mQ'-m'Q)}\Pi(u)$$

for all integral values of m, m'. Then the function $\Pi(u)$ is called a theta function of order r, with the associated constants 2ω, $2\omega'$, 2η, $2\eta'$, and the characteristic (Q, Q'). [A. F. 447, 448.]

It may be proved that the function $\Pi(u)$ exists, and further that if the associated constants and the characteristic are given, there are not more than r^p such functions linearly independent of one another. [A. F. 448–452.]

We notice that the essential character of $\Pi(u)$ is unchanged if a linear transformation be made on the variables u, or if it is multiplied by an exponential factor of the type e^{au^2}, where a is a symmetrical matrix.

The limit to the number of linearly independent functions of order r may be reduced if $\Pi(u)$ is an even function or an odd function of its arguments taken together. In this case it is not difficult to prove that the constants Q, Q' must be half integers [A. F. 462], and the results are:

If $\Pi(-u) = \varepsilon\Pi(u)$, where $\varepsilon = \pm 1$, and r is even, whilst (Q, Q') consists of integers, the number of linearly independent functions $\Pi(u)$ is

$$\leq \tfrac{1}{2}r^p + 2^{p-1}\varepsilon.$$

When r is odd, or when r is even and the characteristic (Q, Q') does not consist wholly of integers, then the number of linearly independent functions is

$$\leq \tfrac{1}{2}r^p + \tfrac{1}{4}[1-(-1)^r]\varepsilon e^{4\pi i QQ'}. \quad \text{[A. F. 463.]}$$

Now suppose θ to be a function of the first order, with half-integer characteristic (q, q'). Then from the properties of such functions we have the result that $e^{4\pi i qq'} = \varepsilon$. [A. F. 251.]

If $\Pi(u)$ is θ^r, it is clear that Π is of the r-th order, with characteristic (rq, rq'). Hence, for functions with the same defining properties as θ^r the above numbers become $\frac{1}{2}r^p + 2^{p-1}$ if r is even, and $\frac{1}{2}(r^p+1)$ if r is odd.

In particular we note that if $r=2$, $\theta(u+v)\theta(u-v)$ has the same defining properties as θ^2, and hence there must be a linear relation, with coefficients independent of u, connecting 2^p+1 of these functions for 2^p+1 different values of the arguments v. There is a similar result for functions of the type $\theta(u-v)\theta(u-w)\theta(u+v+w)$ when $r=3$, and so on for higher values of r.

Now if $F(u)$ is a function multiply periodic in ω, ω', *i. e.*, one such that

$$F(u+\Omega_m)=F(u),$$

for all values of the integers m, m', and if it is made integral by being multiplied by some power, s, of θ, it is clear that if $r>s$, $\theta^r F(u)$ has the same defining properties as θ^r, and hence there is a linear relation among either $\frac{1}{2}r^p+2^{p-1}$ or $\frac{1}{2}(r^p+1)$ such functions, according as r is even or odd, provided $F(u)$ is even.

The second derivatives of $\log\theta$ are such multiply periodic functions. BAKER, BOLZA and others use the notation $\wp_{ij}(u)=-\dfrac{\partial^2}{\partial u_i\,\partial u_j}(\log\theta)$. [See, *e. g.*, A. F. 292, etc.] We shall find it convenient to use (ij) for this function, and similarly, in general,

$$-\frac{\partial^k}{\partial u_r\,\partial u_s\,\partial u_t\ldots\ldots}(\log\theta)\text{ is written }(rst\ldots\ldots);$$

and $(rst\ldots\ldots)$ is a multiply periodic function, which is made integral on multiplication by θ^k.

We shall assume in the remainder of this paper that there is no polynomial relation of either the first or second order connecting the derivatives (ij). Such a relation would in fact be a limitation on the generality of the constants in ω, ω'. For example, if $p=2$, we could reduce a linear relation among (11), (12), (22) to the form $(12)=0$, and θ would reduce to the product of two elliptic theta functions. The particular cases for which such a relation exists are of some interest when $p=3$. We propose to consider them in a later paper.

§ 2.

We consider first the case $p=2$. In this case there are four linearly independent functions of the second order with the same period properties as θ^2. They are θ^2, $\theta^2(11)$, $\theta^2(12)$, $\theta^2(22)$. It is easy to verify that $[(11)(22)-(12)^2]\,\theta^3$ is integral, and hence the five functions with the same period properties as θ^3 are

$$\theta^3\{[(11)(22)-(12)^2],\ (11),\ (12),\ (22),\ 1\}.$$

To save repetition we shall say that a function $f(u)$ is of the r-th order when $f(u)$ is even, multiply periodic, and $\theta^r f(u)$ is integral. It is clear that a function of the r-th order is also a function of the s-th order if r is less than s.

The functions of the fourth order, ten in number, are

$$1,\ (11),\ (12),\ (22), \text{ and all products of the type } (pq)(rs).$$

Of the sixth-order functions twenty are linearly independent. Now they include I) $[(11)(22)-(12)]^2$, II) all products $(pq)(rs)(tu)$, III) the functions of the fourth order.

These are in number 21, and therefore they must be connected by a linear relation. We thus see *a priori* that there is a quartic relation among the three second derivatives (11), (12), (22). This turns out to be Kummer's Quartic Surface.

Again (pqr) is of the third order, except that it is an odd function, and hence any product $(pqr)(stu)$, being of the sixth order, must be expressible as a cubic polynomial in (11), (12), (22). These considerations are useful as showing the kind of relations we are to expect. To obtain them we make use of the fact that

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)}$$

is a function of the second order for all values of the variables v, and hence if it is expanded in powers of the v's, all its coefficients are such functions. We write $\theta = e^{-f}$; then

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)} = exp\left\{-2\left[\frac{1}{\lfloor 2}\left(v\frac{\partial}{\partial u}\right)^2 f + \frac{1}{\lfloor 4}\left(v\frac{\partial}{\partial u}\right)^4 f + \dots\right]\right\}.$$

If the right-hand side be expanded, the coefficients of products of the fourth and sixth orders of v are readily obtained, and we have

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)],$$
$$(pqrstu) - 2\Sigma(pq)(rstu) + 4\Sigma(pq)(rs)(tu),$$

for these coefficients, where the summations extend to all possible combinations of the six letters p, q, r, s, t, u. These expressions are therefore both of them functions of the second order.

Hence we must have $(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] =$ a linear function of 1, (11), (12), (22), $= \sum_{h,k} b_{pqrs}^{(hk)}(hk) + b_{pqrs}$, say, where the b's are constants, and the summation extends once to each pair of values of h, k. Now, by giving p, q, r, s the values 1, 2 we obtain five such equations. These are differential equations of the fourth order for a single function f, and their coexistence by no means follows for general values of the constants b.

In fact, if we differentiate the five equations once, and eliminate fifth derivatives, we obtain four equations among third and second derivatives; these are homogeneous and linear in the third derivatives, which may therefore be eliminated. We thus have an equation among second derivatives only. We might use this equation to obtain by differentiation other homogeneous linear equations in third derivatives, and then by elimination other equations connecting the second derivatives. It is simpler, however, to differentiate the five fundamental differential equations twice and then to eliminate from them the sixth derivatives that occur. We thus get eight equations involving second derivatives and certain functions of third derivatives of the type $(pqr)(hks)-(pqs)(hkr)$. There are only three of these latter functions, and thus by elimination we get five equations which turn out to be of the form

$$A_i[(11)(22)-(12)^2]+B_i(11)+C_i(12)+D_i(22)+E_i=0 \quad (i=1, 2, \ldots, 5),$$

where A, B, C, D, E are constants. As we have assumed that no such relation exists, A, B, C, D, E must all be zero.

If we denote the right-hand side of the typical fundamental equation by B_{pqrs}, and additional suffixes denote differentiations, and if $B_{pqr\alpha,\, s\beta}-6(s\beta)B_{pqr\alpha}$, after fourth derivatives have been replaced by their values in terms of second derivatives, is written $[pqr\alpha, s\beta]$, it is not difficult to see that a typical one of the equations just mentioned is

$$\begin{aligned}[pqr\alpha, s\beta]+[pq\alpha s, r\beta]+[p\alpha rs, q\beta]+[\alpha qrs, p\beta] \\ =[pqr\beta, s\alpha]+[pq\beta s, r\alpha]+[p\beta rs, q\alpha]+[\beta qrs, p\alpha].\end{aligned}$$

The five equations are therefore

$$[1111, 12]=[1112, 11], \quad [1112, 22]=[1222, 11],$$
$$[1111, 22]+2[1112, 12]=3[1122, 11],$$

and two similar equations obtained by interchanging 1 and 2. We take the second-degree terms in these equations first. $[1111, 12]=[1112, 11]$ becomes, on expansion,

$$(2b_{1111}^{(12)}+4b_{1112}^{(22)})[(11)(22)-(12)^2]+\text{linear terms}=0.$$

Thus we have $b_{1111}^{(12)}+2b_{1112}^{(22)}=0$.

Similarly, from the second equation we have $b_{1112}^{(11)}=b_{1222}^{(22)}$, and from the third $b_{1111}^{(11)}-b_{1112}^{(12)}=3b_{1122}^{(22)}$.

We thus have five relations among the constants b. If these are satisfied, it appears that the remaining relations obtained from the above five equations determine the values of the constants b_{pqrs} uniquely, and otherwise lead to no new relations.

Now, the multiplication of θ by an exponential factor e^{au^2} does not alter its essential nature, and is equivalent to giving arbitrary additive constants to the derivatives (11), (12), (22). If we thus modify our equations, we may make

$$2b_{1111}^{(11)} + b_{1112}^{(12)} = 0, \quad 2b_{2222}^{(22)} + b_{1222}^{(12)} = 0, \quad 2b_{1112}^{(11)} + b_{1122}^{(22)} = 0,$$

and then the fundamental equations take precisely the form given by Baker, "Multiply Periodic Functions," p. 49.

We can now verify without trouble that the equations are compatible, obtain the Kummer and Weddle Surfaces, and obtain the relations for products of third derivatives, exactly as in Baker. We are, however, using this case as an illustration of the method for $p = 3$, and hence shall indicate in outline another method for getting $(pqr)(stu)$ and the Kummer Surface.

We use Baker's notation for the coefficients of the differential equations for fourth derivatives, so that

$$\begin{aligned}
-\tfrac{1}{3}B_{1111} &= a_0a_4 - 4a_1a_3 + 3a_2^2 + a_2(11) - 2a_1(12) + a_0(22),\\
-\tfrac{1}{3}B_{1112} &= \tfrac{1}{2}(a_0a_5 - 3a_1a_4 + 2a_2a_3) + a_3(11) - 2a_2(12) + a_1(22),\\
-\tfrac{1}{3}B_{1122} &= \tfrac{1}{6}(a_0a_6 - 9a_2a_4 + 8a_3^2) + a_4(11) - 2a_3(12) + a_2(22),\\
-\tfrac{1}{3}B_{1222} &= \tfrac{1}{2}(a_1a_6 - 3a_2a_5 + 2a_3a_4) + a_5(11) - 2a_4(12) + a_3(22),\\
-\tfrac{1}{3}B_{2222} &= a_2a_6 - 4a_3a_5 + 3a_4^2 + a_6(11) - 2a_5(12) + a_4(22).
\end{aligned}$$

It is interesting to notice the covariantive form of the equations. If we take du_1 and du_2 as variables x_1 and x_2, then any linear transformation on the u's is equivalent to the same linear transformation on the x's; it is clear that $d^4\wp$ and $d^2\wp$,

$$\equiv (11)x_1^2 + 2(12)x_1x_2 + (22)x_2^2, \quad \equiv a_x^2 \equiv b_x^2 \equiv c_x^2 \equiv \ldots,$$

are invariant under such a transformation. Also

$$(a_0, a_1, a_2, \ldots, a_6)(x_1x_2)^6 \equiv \alpha_x^6 \equiv \beta_x^6 \equiv \ldots$$

is an associated sextic, and our fundamental equations are given by

$$d^4\wp - 6(d^2\wp)^2 = -3(\alpha a)^2 a_x^4 - \tfrac{3}{2}(\alpha\beta)^4 \alpha_x^2 \beta_x^2, \equiv B, \text{ say.}$$

From the coefficient, already given, of v^6 in the original expansion we have $d^6\wp - 30d^2\wp\, d^4\wp + 60(d^2\wp)^3 =$ a homogeneous sextic in du_1, du_2, of which the

coefficients are functions of the second order. If we put this equal to X, X must have its coefficients linear functions of the second derivatives. Now $d^4\wp - 6(d^2\wp)^2 = B$, and hence on differentiation $d^6\wp - 12 d^2\wp d^4\wp - 12(d^2\wp)^3 = d^2B$.

If we eliminate $d^6\wp$ and $d^4\wp$ from these equations, we have

$$12\{(d^3\wp)^2 - 4(d^2\wp)^3\} + d^2B - 18Bd^2\wp = X.$$

Now we have already indicated the method of obtaining expressions $(pqr)(hks) - (pqs)(hkr)$, and therefore we can obtain the various products $(pqr)(hks)$ by equating cofficients of powers and products of du_1, du_2, in the above equation, as soon as we know the coefficients of X. For example,

$$\begin{aligned}(111)^2 - 4(11)^3 + 3[a_2(11) - 2a_1(12) + a_0(22)](11)\\ + a_0[(11)(22) - (12)^2] = \lambda(11) + \mu(12) + \nu(22) + \rho,\end{aligned}$$

where λ, μ, ν, ρ are constants to be determined.

There are two methods for determining the unknown constants. In the first method we differentiate twice and eliminate fourth and fifth derivatives by means of the fundamental fourth-order equations. We also substitute for the various squares and products of third derivatives that occur. We then remain with an equation which turns out to be in some cases of the second degree, in others of the fourth degree in second derivatives. Now those of the second degree must vanish identically, and by equating their coefficients to zero we have enough equations to determine the unknown constants. If these are substituted in the fourth-degree equations obtained, they all reduce to one and the same equation, which is that of Kummer's Quartic Surface.

In the second method we differentiate the fundamental equations, and by subtraction eliminate fifth derivatives. We thus remain with an equation linear in third derivatives. Such an equation, for example, is

$$6(12)(111) - 6(11)(112) = 3a_3(111) - 9a_2(112) + 9a_1(122) - 3a_0(222).$$

We multiply these equations by the various third derivatives, and substitute for the squares and products of third derivatives their expressions in terms of second derivatives. As before, we get equations either of the second or of the fourth degree in second derivatives, and again we can determine the unknown constants and obtain Kummer's Quartic. In either case the work may be somewhat simplified, if we notice that $d^2B - 6Bd^2\wp = 4a_x^6[(11)(22) - (12)^2]$ + a quantity

linear in second derivatives, by incorporating this linear quantity into X. The final result in symbolic form is

$$(d^3\wp)^2 - 4(d^2\wp)^3 = -\tfrac{1}{2}(ab)^2\alpha_x^6 - 3(\alpha a)^2\alpha_x^4 b_x^2 - 9(\alpha a)^2(\alpha\beta)^2\alpha_x^2\beta_x^4$$
$$+ \tfrac{9}{2}(\alpha a)(\beta a)\alpha_x^3\beta_x^3 + \tfrac{27}{8}(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2\alpha_x^2\beta_x^2\gamma_x^2.$$

The Kummer Quartic is an invariant of α_x^6 and a_x^2. Its symbolic expression is

$$4(ab)^2(cd)^2 - 16(\alpha a)^2(\alpha b)^2(\alpha c)^2 + 6(\alpha a)^2(\beta b)^2(\alpha\beta)^4 + (ab)^2(\alpha\beta)^6$$
$$+ 9(\alpha a)^2(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^4 - \tfrac{9}{8}(\beta\gamma)^2(\gamma\alpha)^2(\alpha\beta)^2(\alpha\delta)^2(\beta\delta)^2(\gamma\delta)^2 = 0.$$

If we write $d^3\wp, \equiv (111)x_1^3 + \ldots, \equiv p_x^3 \equiv q_x^3 \equiv r_x^3 \equiv \ldots,$ we have for the symbolic equation of the Weddle Surface

$$(hp)^2(hq)(pq)(qr)^2 = 0 \quad [\text{where } h_x^3 \equiv (\alpha p)^3\alpha_x^3].$$

In the above work we have assumed that there exists no quadratic relation among (11), (12), (22); it is interesting to note that the assumption of the existence of such a relation leads to a linear relation among these quantities. By a proper choice of variables such a linear relation could be reduced either to $(11) = 0$, or to $(12) = 0$.

The latter shows that θ must be the product of two elliptic θ functions, whilst the former implies that θ is the product of an elliptic θ and an exponential, e^{Au_1+B}, where A and B are constants.

§ 3.

We now consider the case of $p = 3$. We shall find it convenient to use Δ to denote the determinant

$$\begin{vmatrix} (11), & (12), & (13) \\ (21), & (22), & (23) \\ (31), & (32), & (33) \end{vmatrix}$$

and Δ_{rs} to denote the cofactor of (rs) in Δ.

It may be verified without difficulty that Δ, though apparently of the sixth, is really of the fourth order, whilst Δ_{rs} is of the third order.

In this case there are eight functions of the second order. Of these we have seven, the quantities 1, (11), (12), (13), (22), (23), (33). There must be one linearly independent of these, and this one we call Y. It is to be noticed that for simplifying our equations we may modify Y by adding to it any linear function of the known second order functions (pq) and 1.

The number of linearly independent third-order functions is $\frac{1}{2}(3^3+1), = 14$. We have the eight already given, and the six functions Δ_{rs}. We see at once that there are two cases according as these are or are not linearly independent. As we are assuming that no quadratic relation exists among the quantities (pq), we see that if these 14 third-order functions are not independent, there is a relation

$$Y = \Sigma a_{pq}\Delta_{pq} + \Sigma b_{pq}(pq) + c,$$

where the a's, b's and c are constants. This turns out to be the hyperelliptic case.

There are $\frac{1}{2}4^3 + 4 = 36$ functions of the fourth order. Now we have already eight of these, namely the second-order functions. In addition we must have all products $(pq)(rs)$, $(pq)Y$, Y^2, and Δ. These are in all 37, and hence they must be connected by at least one linear relation. In the hyperelliptic case this relation is the one given above. In the other we shall show later that it may be reduced to the form

$$Y^2 + 2\Delta = \text{a quadratic in the second derivatives } (pq).$$

The number of sixth-order functions is $\frac{1}{2}6^3 + 4 = 112$. We have

I)	21 products	$\Delta_{pq}\Delta_{rs}$,
II)	56 "	$(pq)(rs)(tu)$,
III)	21 "	$Y(pq)(rs)$,
IV)	6 "	$Y^2(pq)$,
V)	21 "	$(pq)(rs)$,
VI)	6 "	$Y(pq)$,
VII)	the 11 functions	(pq), $Y\Delta$, Y^3, Y^2, Y, 1,

that is to say, 142 such functions. These must therefore be connected by 30 linear relations.

In the hyperelliptic case we may neglect 6 of III) which reduce to combinations of I), II) and V); and if we limit ourselves to functions of the fourth or less degree in the derivatives we may neglect IV), $Y\Delta$, Y^3. Also we may neglect VI), Y^2, Y. We thus have 120 functions of degree not greater than four in the second derivatives. They must therefore be connected by at least eight linear relations. In fact it will appear later that there are fifteen such

linearly independent relations. (These relations must of course be connected. In fact, if we use the second derivatives as coordinates in space of six dimensions, we have eight five-folds which pass through a common three-fold. This three-fold is of the eighth order.)

In the non-hyperelliptic case we may neglect IV), which may be expressed in terms of I), II), etc., by the fourth-order relation, and similarly we may neglect Y^3 and Y^2. We thus have 134 functions, and they are at most linear in Y. There must thus be 22 relations of the type $YQ_\iota + A_\iota\Delta = K_\iota$, where Q_ι is quadratic, K_ι quartic in the second derivatives and A_ι is a constant. By consideration of the functions of the fifth order we can show that six of these relations must be of the form $YQ_\iota = C_\iota$, where C_ι is a cubic, and the quadratic Q_ι is linear in the quantities Δ_{pq}, (pq).

Again, it may easily be shown that $R_{pq}, \equiv Y_{pq} - 6(pq)Y$, is of the third order, and therefore, if the case is not hyperelliptic, R_{pq} = a linear function of Δ_{pq}, (pq).

In the hyperelliptic case it appears that this is not true; in fact R_{pq} = a linear function of Δ_{pq}, (pq), J; where J is a certain function cubic in the derivatives, and J occurs in at least one of the expressions.

We now determine the equations in detail. In the first place

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)]$$

is of the second order, and is therefore a linear function of 1, (pq), Y. We thus have fifteen equations of the type

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] = B_{pqrs} + a_{pqrs}Y, \qquad [p, q, r, s = 1, 2, 3], \tag{1}$$

where $B_{pqrs} = \Sigma\, b_{pqrs}^{(ij)}(ij) + b_{pqrs}$, the summation being taken once for each pair of values i, j, and the a's and b's are constants.

If this be differentiated with respect to the variables with suffixes t and u, we obtain

$$\begin{aligned}(pqrstu) - 2\{&(pqtu)(rs) + (pq)(rstu) + (prtu)(qs) + (pstu)(qr)\\ &+ (pr)(qstu) + (ps)(qrtu) + (pqt)(rsu) + (pqu)(rst) + (prt)(qsu)\\ &+ (pru)(qst) + (pst)(qru) + (psu)(qrt)\} = B_{pqrs,\,tu} + a_{pqrs}Y_{tu}.\end{aligned}$$

If s, t are interchanged we obtain another such equation. From these two the sixth derivative may be eliminated by subtraction. The result is

$$2\{(pqu, r)_{st} + (pru, q)_{st} + (qru, p)_{st} + (pq, ru)_{st} + (pr, qu)_{st} + (qr, pu)_{st}\} \\ = B_{pqrs, tu} - B_{pqrt, su} + a_{pqrs} Y_{tu} - a_{pqrt} Y_{su},$$

where

$$(pqu, r)_{st} \equiv (pqus)(rt) - (pqut)(rs),$$
$$(pq, ru)_{st} \equiv (pqs)(rut) - (pqt)(rus).$$

If we permute the suffixes p, q, r, u, we obtain four such equations. They involve the three third-derivative expressions $(pq, ru)_{st}$ and certain fourth and second derivatives. By adding we eliminate the third derivatives, and then by means of (1) we may express fourth derivatives in terms of second derivatives and Y. We thus obtain the equation

$$[pqrs, ut] - [pqrt, us] + a_{pqrs} R_{ut} - a_{pqrt} R_{us} + \text{three similar expressions} \\ \text{obtained by interchanging } u \text{ with } p \text{ and with } q \text{ and with } r = 0, \qquad (2)$$

where $[pqrs, ut] = B_{pqrs, ut} - 6(ut) B_{pqrs}$, in which the fourth derivatives have been replaced by their values from (1).

In addition the four equations mentioned above serve to determine the quantities $(qr, pu)_{st}$. We have in fact

$$4(qr, pu)_{st} + 4\{[(uq)(sp) + (us)(pq)](rt) + [(ur)(sp) + (us)(rp)](qt) \\ - [(uq)(tp) + (ut)(pq)](rs) - [(ur)(tp) + (ut)(rp)](qs)\} + 4Y\{(rt) a_{pqsu} \\ - (rs) a_{pqtu} + (qt) a_{prsu} - (qs) a_{prtu} - (pt) a_{qrsu} + (ps) a_{qrtu} - (ut) a_{pqrs} \\ + (us) a_{pqrt}\} + 4\{(rt) B_{pqsu} - (rs) B_{pqtu} + (qt) B_{prsu} - (qs) B_{prtu} \\ - (pt) B_{qrsu} + (ps) B_{qrtu} - (ut) B_{pqrs} + (us) B_{pqrt}\} \\ = a_{pqrs} R_{tu} - a_{pqrt} R_{su} + a_{qrsu} R_{tp} - a_{qrtu} R_{sp} \\ + [pqrs, tu] - [pqrt, su] + [qrsu, tp] - [qrtu, sp]. \qquad (3)$$

It is worthy of remark that the above equations (2), (3) are of the same general form for any number of variables, for any value of p, the only difference being that if p is greater than three there is more than one function Y.

The equations (2) are linear in the quantities $R_{\alpha\beta}$, $\Delta_{\alpha\beta}$, $(\alpha\beta)$, Y. If we regard $R_{\alpha\beta}$ and Y as unknowns, there are different cases according as they can be solved for some or all of these variables or not. Suppose first that they can be solved for Y; then they give Y as a linear function of Δ_{pq}, (pq). By making

a convenient linear transformation on the fundamental variables this may be modified into one of the forms

$$Y = \Delta_{11} + \Delta_{22} + \Delta_{33}, \quad Y = \Delta_{23}, \quad Y = \Delta_{11}.$$

In this paper we neglect the second two forms, which lead to less general cases than the first, and assume that

$$Y = \Delta_{11} + \Delta_{22} + \Delta_{33}.$$

If we calculate R_{pq} directly from this value of Y, and then use (1), (3) to eliminate third and fourth derivatives, we obtain equations which show that R_{pp} can not be a linear function of the quantities Δ_{rs}, (rs), and from them we deduce without difficulty that

$$a_{pppp} = 3, \quad a_{ppqq} = 1, \quad a_{pppq} = 0, \quad a_{ppqr} = 0, \qquad (p \neq q \neq r).$$

If we multiply the 15 equations (1) by du_1^4, etc., and add, we obtain the equation

$$d^4\wp - 6(d^2\wp)^2 = B + 3C^2 Y,$$

where B is a homogeneous quartic in the differentials du, with coefficients linear in the second derivatives (pq), and C is $du_1^2 + du_2^2 + du_3^2$.

When the particular values given above for the a's are substituted in the equations (2), it appears that they may be solved for the five magnitudes

$$R_{11} - R_{22}, \quad R_{11} - R_{33}, \quad R_{23}, \quad R_{31}, \quad R_{12}.$$

Accordingly each of these five must be expressible as a linear function of the quantities Δ_{pq}, (pq). We therefore assume linear functions of this type, with unknown coefficients, for these magnitudes and substitute their values and that of Y in the equations (2). These equations are now quadratics in second derivatives only, and hence they must vanish identically. Hence by equating their coefficients to zero we obtain equations, which are in fact all that exist among the undetermined constants.

Now the complete determination of the coefficients would be somewhat long if we introduced no further restriction on our choice of variables u_1, u_2, u_3. We notice, however, that Y is an algebraic invariant of the ternary forms $d^2\wp$ and C. In fact, if $d^2\wp = a_x^2 = b_x^2 = \ldots$, and $C = A_x^2 = B_x^2 = C_x^2 = \ldots$, where x is du, we have the relation $3Y(ABC)^2 = (Aab)^2$.

There is thus a certain amount of freedom at our disposal which may be used to get the equations in canonical form. We may, for example, perform any linear transformation on the variables u_1, u_2, u_3, that leaves the quadric C invariant. Also we may add constants to the second derivatives (pq), provided we subtract an appropriate linear function of the second derivatives from Y. Further, it is suggested that the equations can be modified so as to be covariants of certain ternary forms. It appears that the constants to be added to (pq) are uniquely determinate if the equations are to be covariant, and hence that the covariant canonical form is perfectly definite.

The details of the process I adopted are as follows: I first wrote out the general equations among the constants, and then proved that by a certain transformation on the u's the coefficients of type $b^{(33)}_{112}$ and certain constants in the R's could all be made zero. When this simplification was introduced, the constants were all readily calculated in terms of six left arbitrary, and the equations were then in a canonical, though not covariant shape. It was clear, however, that by slight modification they might be made covariantive if a certain fixed quartic were associated, and the proper modification was given without much trouble by comparing terms in B that involved second derivatives with possible covariants.

The fixed forms entering into the covariant equation are the conic C and a certain quartic. The quartic is trinodal, and its nodes are at the vertices of a self-conjugate triangle of the conic. It is clear that by taking C in the form $x_1x_3 - x_2^2$, introducing a parametric pair t_1, t_2, and writing $x_1 = t_1^2$, $x_2 = t_1t_2$, $x_3 = t_2^2$, we may make use of binariants involving a single octavic, that cut out on the conic by the quartic. This leads to a set of equations equivalent to the one given by Baker in the papers already quoted, and serves to identify our equations with the hyperelliptic case, for which a binary octavic is fundamental. I prefer, however, to keep C in the form $x_1^2 + x_2^2 + x_3^2$, so as to preserve symmetry, and to work with orthogonal invariants of a quartic. The quartic is taken to be

$$6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3 \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \gamma_x^4 \equiv \ldots .$$

It involves only five constants, namely the ratios of the six quantities h_1, h_2, h_3, p_1, p_2, p_3. This was to be expected, since the hyperelliptic functions for $p = 3$ possess only five class-moduli. In the non-hyperelliptic case we expect six essential constants.

The values of the coefficients of B for the canonical form are the following:

$$B_{1111}=6(h_2+h_3)(11)+6h_2(22)+6h_3(33)-12p_1(23)+b_{1111},$$
$$B_{1112}=3p_3[(11)+(22)]-6p_2(23)+b_{1112},$$
$$B_{1122}=(h_1+h_2-h_3)[(11)+(22)]-4h_3(33)+2p_1(23)+2p_2(13)+b_{1122},$$
$$B_{1123}=-p_1[2(11)+3(22)+3(33)]-2(h_2+h_3-h_1)(23)$$
$$+2p_2(12)+2p_3(13)+b_{1123},$$

where

$$b_{1111}=3(L+M)-6p_1^2,$$
$$b_{1112}=3p_3(h_2+h_3-h_1)-3p_1p_2,$$
$$b_{1122}=(L-M)-(h_3^2+2h_1h_2-h_1^2-h_2^2)-2p_3^2,$$
$$b_{1123}=-2p_1(2h_1+h_2+h_3)+p_2p_3,$$

and

$$4L=2(h_2h_3+h_3h_1+h_1h_2)-h_1^2-h_2^2-h_3^2,$$
$$M=p_1^2+p_2^2+p_3^2,$$

and the remainder of the B's are obtained by interchanging the suffixes 1, 2, 3.

We can identify the equations with an appropriate symbolic expression by making use of the fact that orthogonal invariants consist of sums of products of symbolic expressions of the types

$$(\alpha_1^2+\alpha_2^2+\alpha_3^2),\quad (\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3),\quad \alpha_x,\quad (\alpha\beta\gamma),$$

and then introducing the fundamental conic to obtain the regular ternary three-rowed determinants. We thus obtain the result

$$d^4\wp-6(d^2\wp)^2=6(A\alpha a)^2\alpha_x^2B_x^2+4(AB\alpha)(ABa)\alpha_x^3a_x-3(ABa)^2\alpha_x^4+3YA_x^2B_x^2+Q,$$

where

$$Q=-\tfrac{1}{8}(A\alpha\beta)^2(B\alpha\beta)^2C_x^2D_x^2+2(A\alpha\beta)^2(B\alpha\beta)(BC\beta)C_xD_x^2\alpha_x-\tfrac{1}{2}(AB\beta)^2(CD\beta)^2\alpha_x^4$$
$$-\tfrac{1}{4}(AB\alpha)^2(CD\beta)^2\alpha_x^2\beta_x^2-\tfrac{3}{4}(AB\alpha)(AB\beta)(CD\alpha)(CD\beta)\alpha_x^2\beta_x^2$$
$$+2(AB\beta)^2(CD\alpha)(CD\beta)\alpha_x^3\beta_x.$$

In the course of determining the above equations we also determine the values of R_{23}, R_{31}, R_{12}, $R_{11}-R_{22}$, $R_{11}-R_{33}$. By direct differentiation of Y we can calculate, for example, R_{11}, and thus we have expressions for the quantities $R_{pq}\equiv Y_{pq}-6(pq)Y$ in terms of second derivatives. Now R_{11}, for example, involves the as yet undetermined third-order function J. Let

$$I=\tfrac{1}{2}(A\alpha a)^2(Bab)^2,=\Sigma h_1(11)^2+(h_1+h_2+h_3)\Sigma(11)(22)$$
$$-2\Sigma p_1(23)[(22)+(33)]+4\Sigma p_1(12)(13),$$

then

$$J=2Y[(11)+(22)+(33)]-\Delta+I,$$

and

$$\begin{aligned}
R_{11} = & -4J - 4(h_1 + h_2 + h_3)(\Delta_{22} + \Delta_{33}) - 8h_1\Delta_{11} + 8p_2\Delta_{31} + 8p_3\Delta_{12} \\
& + [2p_1^2 + 8p_2^2 + 8p_3^2 - 14L + 3(h_1^2 + 2h_2h_3 - h_2^2 - h_3^2)](11) \\
& + [8p_1^2 + 4p_2^2 + 14p_3^2 - 8L + (h_2^2 + 2h_3h_1 - h_3^2 - h_1^2)](22) \\
& + [8p_1^2 + 14p_2^2 + 4p_3^2 - 8L + (h_3^2 + 2h_1h_2 - h_1^2 - h_2^2)](33) \\
& - [22p_2p_3 - 4p_1(4h_1 - h_2 - h_3)](23) - [14p_3p_1 + 2p_2(7h_1 - h_2 + h_3)](31) \\
& - [14p_1p_2 + 2p_3(7h_1 + h_2 - h_3)](12) - (h_2 + h_3 - h_1)(h_3 + h_1 - h_2)(h_1 + h_2 - h_3) \\
& + 2p_1^2(7h_1 + 2h_2 + 2h_3) + 2p_2^2(2h_1 + 7h_2 + 2h_3) + 2p_3^2(2h_1 + 2h_2 + 7h_3) \\
& \qquad - 16p_1p_2p_3 .
\end{aligned}$$

Also

$$\begin{aligned}
R_{12} = & \; 4p_3(\Delta_{11} + \Delta_{22}) - 8p_1\Delta_{13} - 8p_2\Delta_{23} - 8h_3\Delta_{12} \\
& + [p_3(h_2 - 5h_1 - h_3) - 2p_1p_2](11) + [p_3(h_1 - 5h_2 - h_3) - 2p_1p_2](22) \\
& + [6p_3h_3 - 6p_1p_2](33) + [4(h_1 + 2h_2 + h_3)p_2 - 2p_1p_3](23) \\
& + [4(2h_1 + h_2 + h_3)p_1 - 2p_2p_3](31) \\
& + [6(L + M) - 4p_1^2 - 4p_2^2 + 4h_3(h_3 - h_1 - h_2)](12).
\end{aligned}$$

These expressions must of course be coefficients of covariants. The determination of the appropriate symbolic expressions involves rather long calculation for the lower-degree terms in second derivatives. We have however the result that

$$\begin{aligned}
\tfrac{1}{6}(ABC)^2[d^2Y - 6Y(d^2\varphi)] = & \; C\{-4J + (\alpha ab)^2(\alpha AB)^2 - (\alpha AB)^2(\alpha CD)^2 Y\} \\
& + 2(Aab)(Bab)(AC\alpha)(BC\alpha)\alpha_x^2 + H + KC,
\end{aligned}$$

where H is a quadratic covariant linear in the quantities (pq), and K is an invariant with constant coefficients.

We next need the expressions, which we know *a priori* to exist, for products of third derivatives as cubic functions of second derivatives. It seems difficult to find these by direct integration, as may be done for $p = 2$, because the complication introduced by the function Y makes the algebra heavy. In the case of $p = 2$ the cubics are minors of a four-rowed determinant. The corresponding determinantal expressions in our case are not at all obvious, for instead of a four-rowed determinant we now have an array with ten rows and twenty-four columns, all of whose ten-rowed determinants must vanish. The third derivatives are proportional to first minors of these determinants, but it would seem impossible to readily factor the nine-rowed determinants, since there are many quartic relations among their elements.

We therefore use the method indicated for $p=2$. We have, as before, the equation

$$d^6\wp - 30\,d^2\wp\, d^4\wp + 60\,(d^2\wp)^3 = 12X,$$

where X is a covariant of the sixth order, with coefficients of the first degree in (pq), Y. Also

$$d^4\wp - 6\,(d^2\wp)^2 = B + 3\,YC^2.$$

Hence, by differentiation

$$d^6\wp - 12\,(d^2\wp)(d^4\wp) - 12\,(d^3\wp)^2 = d^2B - 3\,C^2 d^2 Y,$$

and therefore, by elimination of $d^6\wp$, $d^4\wp$, we have the equation

$$(d^3\wp)^2 - 4\,(d^2\wp)^3 - (B + 3\,YC^2)\,d^2\wp + \tfrac{1}{12}[d^2B - 6\,B(d^2\wp)] \\ + \tfrac{1}{4}C^2[d^2Y - 6\,Y(d^2\wp)] = X. \qquad (4)$$

The terms of the second degree in $d^2B - 6B(d^2\wp)$ may be readily obtained from the symbolic expression for B. They are all linear functions of the quantities Δ_{pq}, (pq), and hence are only of the third order. Also the quantities $d^2Y - 6\,Y(d^2\wp)$ are known, so that the coefficients of the left-hand side of (4) are easily obtained. We can therefore express such functions as (111), (111)(112), $2(111)(122) + 3(112)^2$, etc., by means of quantities of the third degree in second derivatives, and all the coefficients except those arising from X are known. Again, (3) gives, for example, $(111)(122) - (112)^2$ as a third-degree expression, and hence we can find (111)(122) and $(112)^2$. Similarly we can find expressions for all the remaining products of third derivatives, though the coefficients arising from X are as yet undetermined. Now $Y = \Delta_{11} + \Delta_{22} + \Delta_{33}$ and therefore

$$Y_1 = (111)[(22) + (33)] + (122)[(33) + (11)] + (133)[(11) + (22)] \\ - 2(123)(23) - 2(113)(13) - 2(112)(12).$$

We multiply this equation by any third derivative, and substitute the expressions already obtained for products of third derivatives. We thus obtain the values of such products as $(pqr)\,Y_s$ in terms of second derivatives, and in these expressions all the coefficients are known except those of the second and lower degrees and those associated with such functions as $(pq)\,Y$.

We now obtain, by differentiation and subtraction of the fundamental equations (1), 24 equations linear in second derivatives, linear in third derivatives, and possibly containing a term that is a first derivative of Y.

For example, we have

$$2(111)(12) - 2(112)(11) = 2\,(h_2 + h_3)(112) + 2\,h_2(222) + 2\,h_3(233) \\ - 4p_1(223) - p_3[(111) + (122)] + 2p_2(123) + Y_1.$$

We multiply these equations by one of the third derivatives, substitute for products of third derivatives, and obtain finally an equation which is either of the fourth or of the second degree in second derivatives. The fact that the equations of the second degree must vanish identically enables us to determine the unknown constants, and then the remaining equations give the relations of the fourth degree among the second derivatives.

The work indicated above is very long, both on account of the number of functions to be calculated, and on account of the magnitude of the final results. We content ourselves therefore, for the present, with giving the terms of highest degree for the various functions mentioned. The typical products of third derivatives are given by the equations

$$
\begin{aligned}
&(111)^2 - 4(11)^3 - J - 3(11)Y = \ldots, \\
&(111)(112) - 4(11)^2(12) - (12)Y = \ldots, \\
&(111)(122) - 2(11)^2(22) - 2(11)(12)^2 + J - [(11) + 2(22)]\,Y = \ldots, \\
&(112)^2 - 4(11)(12)^2 + Y(22) - J = \ldots, \\
&(111)(123) - 2(11)(12)(13) - 2(11)^2(23) - 2Y(23) = \ldots, \\
&(112)(113) - 4(11)(12)(13) + Y(23) = \ldots, \\
&(111)(222) + 2(12)^3 - 6(11)(22)(12) + 3Y(12) = \ldots, \\
&(112)(122) - 2(11)(22)(12) - 2(12)^3 - Y(12) = \ldots, \\
&(111)(223) - 4(11)(12)(23) - 2(11)(13)(22) + 2(12)^2(13) + Y(13) = \ldots, \\
&(112)(123) - 2(11)(12)(23) - 2(12)^2(13) = \ldots, \\
&(113)(122) - 2(11)(13)(22) - 2(12)^2(13) - Y(13) = \ldots, \\
&(123)^2 - 4(23)(31)(12) + \Delta - Y[(11) + (22) + (33)] = \ldots, \\
&(112)(233) - 2(11)(23)^2 - 2(33)(12)^2 - \Delta + Y(22) = \ldots,
\end{aligned}
$$

where the terms omitted are of the second or lower degree in the second derivatives.

Those for the products $(pqr)Y_s$ are

$$
\begin{aligned}
(111)Y_1 &= Y[4(11)^2 + 4\Delta_{11} + 2\Delta_{22} + 2\Delta_{33}] + \ldots, \\
(222)Y_1 &= Y[4(22)(12) - 2\Delta_{12}] + \ldots, \\
(112)Y_1 &= Y[4(11)(12) + 2\Delta_{12}] + \ldots, \\
(122)Y_1 &= 2Y[(11)(22) + (12)^2 + \Delta_{22}] + \ldots, \\
(123)Y_1 &= 4Y(12)(13) + \ldots, \\
(223)Y_1 &= 2Y[(13)(22) + (12)(23)] + \ldots,
\end{aligned}
$$

where the omitted terms are of the third or lower degree.

Finally, the quartics are

$$\begin{aligned}
&(11)J + Y\Delta_{11} + \ldots\ldots = 0,\\
&(12)J + Y\Delta_{12} + \ldots\ldots = 0,\\
&(\Delta_{11} + \Delta_{22})^2 + [(11) + (22)]^2 Y + \ldots\ldots = 0,\\
&\Delta_{23}^2 + [(11) + (22)][(11) + (33)]Y + \ldots\ldots = 0,\\
&\Delta_{13}\Delta_{23} - (12)[(11) + (22)]Y + \ldots\ldots = 0,\\
&\Delta_{23}(\Delta_{11} + \Delta_{22}) + (23)[(11) + (22)]Y + \ldots\ldots = 0,
\end{aligned}$$

where the terms omitted are of the third or lower degree.

These quartics may obviously be expressed as the coefficients of two covariants, one of the fourth, the other of the second order. It may be proved that of the 21 quartics indicated, 15 are linearly independent. *There are thus 15 linearly independent relations of the fourth degree among the second derivatives of the logarithm of a hyperelliptic theta function of genus three.*

Now these relations can not be functionally independent; it is in fact clear that since the second derivatives are functions of three independent variables, the quartics, regarded as five-fold spreads in space of six dimensions, must all pass through a common three-fold. It is easy to see, by consideration of the highest-degree terms, that the surface at infinity for this three-fold is given by the vanishing of all the first minors of Δ. These six quantities all vanish if any three of them are zero, and it follows that the surface at infinity and therefore the three-fold itself are of the eighth degree.

Thus the generalization of the Kummer Quartic Surface is a certain three-fold spread of the eighth degree in space of six dimensions.

§ 4.

We next consider the non-hyperelliptic case. The function Y is not now a quadratic function of the second derivatives, and thus the third-order functions Δ_{pq}, (pq), Y, 1 are linearly independent. Hence there can be no others linearly independent of these, and therefore the quantities R_{pq} must be linearly expressible in terms of them. We write

$$R_{pq} = \sum_{h,k} c_{pq,hk}\Delta_{hk} + \sum_{h,k} d_{pq}^{(hk)}(hk) + d_{pq} + e_{pq}Y,$$

where the summation extends once to each pair of values of h, k, and substitute in the equation (2). This equation now becomes a linear function of the fourteen third-order functions, and hence it must vanish identically.

By equating its coefficients to zero there is obtained as before a set of relations among the various constants involved. It appears that there are no limitations on the constants a_{pqrs}. We may modify our theta function as before by multiplying by an exponential factor, and also we may add any linear function of the second derivatives to Y. The only additional constants that enter into the equation may be got rid of by these modifications, and therefore the only essential constants are the a's. We find that the modifications above mentioned can be made in only one way (with a trivial exception), if the equations are to be covariantive. In this case there is one fixed form, which is the general quartic

$$F \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \ldots \equiv \sum_{p,q,r,s} a_{pqrs} x_p x_q x_r x_s, \qquad (p, q, r, s = 1, 2, 3).$$

The equations, in symbolic form, are given by

$$d^4\wp - 6(d^2\wp)^2 = YF + (a\alpha\beta)^2 \alpha_x^2 \beta_x^2 - \tfrac{1}{18} S, \tag{5}$$

where

$$S = (\beta\gamma\delta)(\gamma\delta\alpha)(\delta\alpha\beta)(\alpha\beta\gamma)\,\alpha_x\beta_x\gamma_x\delta_x;$$

and

$$d^2Y - 6Yd^2\wp = -2(ab\alpha)^2\alpha_x^2 + \tfrac{1}{6}(\alpha\beta a)^2(\alpha\beta\gamma)^2\gamma_x^2 - \tfrac{1}{72}(\alpha\beta\gamma)^2(\alpha\delta\varepsilon)^2(\delta\beta\gamma)(\varepsilon\beta\gamma)\,\delta_x\varepsilon_x. \tag{6}$$

These covariant expressions are definite except in one particular. It is seen at once that Y behaves like an invariant of the third degree. Now $A \equiv \frac{1}{6}(\alpha\beta\gamma)^4$ is a similar invariant, and hence we may, if we like, take instead of Y the function $Y' = Y + \lambda A$, where λ is an absolute constant. There is then a corresponding modification to be made to the second term on the right-hand side of (6). For instance, if $Y' = Y - \frac{1}{18}A$, the identity

$$(\alpha\beta\gamma)^2(\beta\gamma a)^2\alpha_x^2 = 2(\alpha\beta a)(\alpha\gamma a)(\alpha\beta\gamma)^2\beta_x\gamma_x + 2A a_x^2$$

shows that this second term becomes $\frac{1}{3}(\alpha\beta a)(\alpha\gamma a)\beta_x\gamma_x$.

These expressions were obtained by taking the quartic in the canonical form

$$F = x_1^4 + x_2^4 + x_3^4 + 6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3,$$

and calculating the constants directly from the equations among the coefficients of the equations of type (2). It then appeared that the part of B (we recall that B is used to denote that part of the right-hand side of (5) that does not involve Y) involving first derivatives was quadratic in the constants of F. The only available covariant was therefore $(\alpha\beta a)^2\alpha_x^2\beta_x^2$, and it was found that by adding

suitable constants to the second derivatives, and incorporating a linear function of these second derivatives into Y, this part of B could be identified with this covariant. It is interesting to note that the method followed leads to the explicit forms of the various covariants involved, and therefore is one for calculating certain covariants of the quartic. The constant terms are two such covariants for which we thus have explicit expressions. They are both given by Salmon* for the particular case in which p_1, p_2, p_3 are all zero.

We proceed to give the explicit forms of the equations:

$$
\begin{aligned}
B_{1111} &= 2(h_2h_3 - p_1^2)(11) + 2h_2(22) + 2h_3(33) - 4p_1(23) + b_{1111}, \\
B_{1112} &= (h_3p_3 - 2p_1p_2)(11) + p_3(22) - 2(h_2h_3 - p_1^2)(12) - 2p_3(23) + b_{1112}, \\
B_{1122} &= \tfrac{1}{3}(h_2 + h_1h_3 - 4p_2^2)(11) + \tfrac{1}{3}(h_1 + h_2h_3 - 4p_1^2)(22) + \tfrac{1}{3}(1 - 3h_3^2)(33) \\
&\quad + 2h_3p_1(23) + 2h_3p_2(31) + \tfrac{2}{3}(5p_1p_2 - 4h_3p_3)(12) + b_{1122}, \\
B_{1123} &= -\tfrac{2}{3}(h_1p_1 + p_2p_3)(11) - h_2p_1(22) - h_3p_1(33) + \tfrac{2}{3}(2h_2h_3 + p_1^2 - h_1)(23) \\
&\quad + \tfrac{2}{3}(h_3p_3 + p_1p_2)(31) + \tfrac{2}{3}(h_2p_2 + p_3p_1)(12) + b_{1123},
\end{aligned}
$$

where the coefficients of $-S$ are given by

$$
\begin{aligned}
18b_{1111} &= -24(h_2h_3 - p_1^2) - 24p_1p_2p_3 + 24h_2p_2^2 + 24h_3p_3^2, \\
18b_{1112} &= 6\{(h_3p_3 - 2p_1p_2)(p_1^2 - h_2h_3) + 2h_3h_1p_3 - h_1p_1p_2 - h_2p_3\}, \\
18b_{1122} &= 4h_1h_2h_3^2 - 4h_1h_3p_1^2 - 4h_2h_3p_2^2 - 4h_3^2p_3^2 + 16h_3p_1p_2p_3 - 12p_1^2p_2^2 \\
&\quad - 4h_1p_2^2 - 4h_2p_1^2 + 4h_1^2h_3 + 4h_2^2h_3 - 4h_1h_2 - 4p_3^2, \\
18b_{1123} &= -8h_1h_2h_3p_1 - 6h_2h_3p_2p_3 + 8h_1p_1^3 + 8h_2p_1p_2^2 + 8h_3p_1p_3^2 - 14p_1^2p_2p_3 \\
&\quad + 4h_1p_2p_3 - 2h_1^2p_1 + 2p_1.
\end{aligned}
$$

Also

$$
\begin{aligned}
R_{11} &= -4\Delta_{11} - 4h_3\Delta_{22} - 4h_2\Delta_{33} - 8p_1\Delta_{23} \\
&\quad + \tfrac{1}{3}[2h_1h_2h_3 - 2p_1p_2p_3 + 4h_1p_1^2 + 2h_2p_2^2 + 2h_3p_3^2 + 3h_1^2 + h_2^2 + h_3^2 + 1](11) \\
&\quad + \tfrac{1}{3}[4h_2(h_2h_3 - p_1^2) + 2(h_1h_2 + h_3 + p_3^2)](22) + (\bigstar)(33) \\
&\quad + \tfrac{1}{3}[8p_1(h_1 - h_2h_3 + p_1^2) - 2p_2p_3](23) \\
&\quad + \tfrac{1}{3}[6p_1p_3h_3 - 2p_2(h_2h_3 + 2p_1^2 + 3h_1)](31) + (\bigstar)(12) + d_{11}, \\
R_{12} &= -4p_3\Delta_{33} - 8p_2\Delta_{23} - 8p_1\Delta_{31} - 8h_3\Delta_{12} \\
&\quad - \tfrac{1}{3}[5h_1h_3p_3 + 2h_1p_1p_2 - h_2p_3](11) + (\bigstar)(22) \\
&\quad + \tfrac{1}{3}[7h_3(h_3p_3 - 2p_1p_2) + p_3](33) \\
&\quad + \tfrac{1}{3}[4p_2(3h_2h_3 + p_1^2 + h_1) - 2h_3p_1p_3](23) + (\bigstar)(31) \\
&\quad + \tfrac{1}{3}[4h_1h_2h_3 - 10p_1p_2p_3 + 8h_1p_1^2 + 8h_2p_2^2 + 12h_3p_3^2 + 4h_3^2](12) + d_{12},
\end{aligned}
$$

* "Higher Plane Curves," 3rd Ed., pp. 270, 273.

where

$$
\begin{aligned}
9d_{11} = {} & 4h_1h_2^2h_3^2 + 4h_1h_2h_3p_1^2 - 8h_1p_1^4 + 8p_1^3p_2p_3 + 28h_2h_3p_1p_2p_3 - 8h_2p_1^2p_2^2 \\
& - 10h_2^2h_3p_3^2 - 8h_3p_1^2p_3^2 - 10h_2h_3^2p_3^2 - 6h_1^2h_2h_3 - 20h_1p_1p_2p_3 + 6h_1^2p_1^2 \\
& + 8h_1h_2p_2^2 + 8h_1h_3p_3^2 + 6p_2^2p_3^2 - 2h_2^2p_1^2 - 2h_3^2p_1^2 + 2h_2^3h_3 + 2h_2h_3^3 \\
& + 2h_2p_3^2 + 2h_3p_2^2 - 2h_2h_3 + 2p_1^2, \\
9d_{12} = {} & - 8h_1h_2h_3^2p_3 - 2h_1h_2h_3p_1p_2 - 6h_3p_1p_2p_3^2 + 4p_1^2p_2^2p_3 + 2h_1h_3p_1^3p_3 \\
& - 4h_1p_1^3p_2 + 2h_2h_3p_2^2p_3 - 4h_2p_1p_2^3 + 2h_3^2p_3^3 + 2h_1^2h_3p_3 + 2h_2^2h_3p_3 \\
& + 2h_1^2p_1p_2 + 2h_2^2p_1p_2 + 2h_3^3p_3 - 4h_3^2p_1p_2 - 6h_1p_2^2p_3 - 6h_2p_1^2p_3 - 2p_3^3 \\
& + 4h_1h_2p_3 - 2h_3p_3 + 2p_1p_2.
\end{aligned}
$$

The expressions omitted are obtained by appropriate interchange of the suffixes 1, 2, 3.

We shall find it convenient to use the symbolic notation for most of the remainder of our work. We use $p_x^3 \equiv q_x^3 \equiv \ldots.$ for the form whose coefficients are third derivatives, and ξ_x^4 for fourth derivatives; also L_x, M_x^2, N_x^3, ... are used to denote first, second, etc., derivatives of Y.

It is easy to differentiate a symbolic expression. For example, suppose we have a covariant linear in second derivatives, say $(a\alpha\beta)^2\alpha_x^2\beta_x^2$. The first derivative of this is $(p\alpha\beta)^2p_x\alpha_x^2\beta_x^2$; the second is

$$(\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \equiv \xi_y^2\xi_x^2\alpha_x^2\beta_x^2.$$

We now use the polarized form of (5) and can substitute for ξ at once.

The differential coefficient of $(ab\alpha)^2\alpha_x^2$ is $2(pa\alpha)^2p_x\alpha_x^2$. Its second derivative is

$$2(pq\alpha)^2p_xq_x\alpha_x^2 + 2(\xi a\alpha)^2\xi_x^2\alpha_x^2.$$

This may be expressed in terms of second derivatives and Y only, as soon as we know the expression for $(pqu)^2p_xq_x$, that is to say, the expression for (3) in symbols.

We now proceed to get some equations that we need in symbolic form. We first take (5) in completely polarized form, $d^{(1)}d^{(2)}d^{(3)}d^{(4)}\rho = \ldots.$, and perform on it the operation $d^{(5)}$. We then interchange $d^{(4)}$ and $d^{(5)}$ and subtract. We thus obtain an equation which is linear in second derivatives, linear in third derivatives, and linear in first derivatives of Y. This is the polarized form of

$$-6(pau)p_x^2a_x = (p\alpha\beta)^2(p\beta u)\alpha_x^2\beta_x + (Lau)\alpha_x^3, \tag{7}$$

where u has been written for $(x^{(4)}x^{(5)})$. In exactly the same way we derive from (7) the equation

$$4(pqu)^2 p_x q_x = 2(\xi a u)^2 \xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi a u)(\xi\beta u)\alpha_x\beta_x + (Mau)^2\alpha_x^2. \tag{8}$$

If in this we replace fourth derivatives by their values in terms of second derivatives, and substitute from (6) for second derivatives of Y, we have equation (3) given in symbolic form.

We shall need in later work the value of the second derivative of $(abu)(abv)$. This may be obtained by polarizing from $(abu)^2$, and so we give $d^2(abu)^2$.

Now

$$d^2(abu)^2 = 2(pqu)^2 p_x q_x + 2(\xi a u)^2 \xi_x^2,$$

and therefore from (8)

$$2d^2(abu)^2 = 6(\xi a u)^2\xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi a u)(\xi\beta u)\alpha_x\beta_x + (Mau)^2\alpha_x^2.$$

§ 5.

Exactly as in previous cases we have an equation

$$d^6\wp - 30 d^2\wp\, d^4\wp + 60(d^2\wp)^3 = X + hY,$$

where X consists of two parts, one linear in second derivatives, and the other constant, whilst h is constant, both X and h being sextic covariants. As before, we eliminate fourth and sixth derivatives by means of (5) and thus obtain an equation

$$12\left[(d^3\wp)^2 - 4(d^2\wp)^3 - (d^2\wp)(B+FY)\right] + K = X + hY, \tag{9}$$

where

$$K = d^2B - 6Bd^2\wp + F(d^2Y - 6Yd^2\wp).$$

Now

$$B = (a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{const.},$$

therefore

$$\begin{aligned} d^2B &= (\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \\ &= 2(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 4(a\alpha\beta)(b\alpha\beta)a_x b_x \alpha_x^2\beta_x^2 + \tfrac{1}{3}(\alpha\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 \\ &\quad + \tfrac{2}{3}(\alpha\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x + (\sigma\alpha\beta)^2\sigma_x^2\alpha_x^2\beta_x^2 + (\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2\, Y; \end{aligned}$$

and

$$6Bd^2\wp = 6(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 6\sigma_x^4 a_x^2;$$

also

$$d^2Y - 6Yd^2\wp = -2(aba)^2 a_x^2 + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 + \tau_x^2,$$

where

$$\sigma_x^4 = -\tfrac{1}{18} S, \ \tau_x^2 = \text{the constant term in (6).}$$

Hence

$$\begin{aligned} K = & -6\,(ab\alpha)^2 \alpha_x^2 F + 4\,(ab\alpha)\,(ab\beta)\,\alpha_x^3 \beta_x^3 + [\tfrac{1}{3}\,(a\gamma\delta)^2\,(\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \delta_x^2 \\ & + \tfrac{2}{3}\,(a\gamma\delta)^2 (\alpha\beta\gamma)\,(\alpha\beta\delta)\,\alpha_x^2 \beta_x^2 \gamma_x \delta_x + \tfrac{1}{3} S a_x^2 + \tfrac{1}{6}\,(a\alpha\beta)^2 (\alpha\beta\gamma)^2 \gamma_x^2 F] \\ & + [\sigma\alpha\beta)^2 \sigma_x^2 \alpha_x^2 \beta_x^2 + \tau_x^2 F] + Y(\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \gamma_x^2 \\ = & \ P + Q, \text{ say,} \end{aligned}$$

where

$$P = -6\,(ab\alpha)^2 \alpha_x^2 F + 4\,(ab\alpha)\,(ab\beta)\,\alpha_x^3 \beta_x^3 + HY,$$

and

$$H = (\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \gamma_x^2 .$$

Thus when we know the quantities X and h, the equations (8), (9) serve to express products of third derivatives as cubic functions of second derivatives, and involving Y linearly. These equations, however, enable us to get the expression for Y in terms of second derivatives. To obtain this relation we differentiate (9) twice, and substitute from (8) for the third derivatives that occur. We thus obtain a relation of the form

$$(Y^2 + 2\,\Delta)\,F^2 = \Sigma\alpha_{pq,\,rs}\,(pq)(rs) + \Sigma\beta_{pq}\,(pq) + \gamma + \delta Y,$$

where the coefficients on the right side are constants. It is clear that F^2 must divide through the equation, and thus we have restrictions on X and h, which enable us to determine these quantities, with the exception of the constant term in X.

When we differentiate (9) twice and substitute for fourth derivatives, we have the equation

$$12\,(B + YF)^2 + d^2K - 12\,Kd^2\wp = d^2X + hd^2Y.$$

We can calculate d^2K in symbols. Also, X being a covariant at most linear in the coefficients of a_x^2, we see that $d^2X - 6\,Xd^2\wp$ is an expression linear in the quantities Δ_{pq}, (pq). When the above expression is written in expanded form, it appears that F^2 is a factor of all terms except those of the types $(pq)(rs)$, $(pq)\,Y$, (pq), 1. It must therefore also divide the terms of the types given. It follows at once that there are no terms of the type $(pq)\,Y$, and hence h is determined. A consideration of terms of the type $(pq)(rs)$ determines the part of X involving the coefficients of a_x^2. We obtain finally the result

$$\begin{aligned} 12\,[(d^3\wp)^2 - 4\,(d^2\wp)^3 - (d^2\wp)(B + FY)] + K = & -\tfrac{5}{3}\,HY + Z \\ & - 5\,(a\gamma\delta)^2 (\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \delta_x^2 + 6\,(a\gamma\delta)^2 (\alpha\beta\gamma)(\alpha\beta\delta)\,\alpha_x^2 \beta_x^2 \gamma_x \delta_x - \tfrac{1}{3} S a_x^2 \\ & + \tfrac{10}{3} A F a_x^2 + \tfrac{5}{3} F (a\alpha\beta)^2 (\alpha\beta\gamma)^2 \gamma_x^2 - 4\,(a\alpha\beta)(\delta\alpha\beta)(\alpha\beta\gamma)^2 a_x \gamma_x^2 \delta_x^3 \ \ldots\ldots \text{(10)} \end{aligned}$$

where

$$H = (\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \gamma_x^2,$$

and Z is a sextic covariant of the quartic alone, of the sixth order in the coefficients.

This equation (10), in conjunction with (8), gives expressions for all squares and products of third derivatives as cubic functions of the second derivatives, and involving Y linearly. The constant terms, the coefficients of Z, are however as yet undetermined.

The resulting equation in the above work, when F is divided out, is an expression for Y as a function of the second derivatives. This equation is

$$Y^2 - \tfrac{1}{18} A Y = -2\Delta + \tfrac{1}{6}(a\alpha\beta)^2(b\alpha\beta)^2 + \tfrac{1}{72}(a\alpha\beta)^2(\alpha\gamma\delta)^2(\beta\gamma\delta)^2 - \tfrac{5}{7} D, \quad (11)$$

where D is the six-rowed determinant, the invariant of the quartic of the sixth degree in the coefficients, given by Salmon, "Higher Plane Curves," 3rd Ed., p. 265, and called by him B.

This equation was in fact calculated as follows: All the terms except those linear in second derivatives and the constant were determined in the manner indicated above. The remaining terms, being invariant, were necessarily of the form

$$n_1(a\alpha\beta)^2(\alpha\gamma\delta)^2(\beta\gamma\delta)^2 + n_2 A^2 + n_3 D,$$

where n_1, n_2, n_3 were merely numerical constants. n_2 was then shown to be zero by means of the particular quartic $x_1^4 + x_2^4 + x_3^4$, and the remaining two constants were found by calculating out one of the equations for Salmon's form of the quartic, $\Sigma x^4 + 6\,\Sigma h_1 x_2^2 x_3^2$.

There yet remain to be determined the relations connecting second derivatives only. Now there are 70 functions of the fifth order involving second derivatives and Y. One relation is given by (11), and since there are only 63 linearly independent fifth-order functions, it is clear that there must be six other such relations.

We proceed to obtain these relations, to show, in fact, that the coefficients of squares and products of the u's in $Y(abu)^2$ may be linearly expressed by means of the other fifth-order functions.

Equation (6) in polarized form is

$$M_x M_y - 6\,Y a_x a_y = -2(ab\alpha)^2 \alpha_x \alpha_y + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2 \gamma_x \gamma_y + \tau_x \tau_y.$$

We differentiate this with respect to z, then interchange y and z in the equation obtained and subtract; we thus get the equation

$$-6(Lau)\,a_x = -4(paa)^2(pau)\,a_x + \tfrac{1}{6}(pa\beta)^2(\alpha\beta\gamma)^2(p\gamma u)\,\gamma_x \ldots, \qquad (12)$$

where u is written for (yz).

We next differentiate (12) with respect to y, interchange x and y, subtract, and then write $(xy) = u$. We thus have the equation

$$6(Mau)^2 = 4(\xi a\alpha)^2(\xi\alpha u)^2 + 4(pq\alpha)^2(pu\alpha)(qu\alpha) - \tfrac{1}{6}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(u\xi\gamma)^2.$$

In this equation we substitute for $(Mau)^2$ from (6), and for $(pq\alpha)^2(pu\alpha)(qu\alpha)$ from (8), after writing in (8), α for u, and $(u\alpha)$ for x. We thus have the relation

$$\begin{aligned}36\,Y(abu)^2 - 12(ab\alpha)^2(cu\alpha)^2 + (a\alpha\beta)^2(\alpha\beta\gamma)^2(bu\gamma)^2 + 6(\tau a u)^2 \\ = 6(a\xi\alpha)^2(u\xi\alpha)^2 + (\xi\alpha\beta)^2(\xi\beta\gamma)^2(\alpha\gamma u)^2 - \tfrac{1}{2}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(\xi\gamma u)^2 \\ + (M\alpha\beta)^2(\alpha\beta u)^2.\end{aligned}$$

This relation, after substitution for ξ and M, becomes

$$\begin{aligned}36\,Y\big[(abu)^2 - \tfrac{1}{3}(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{72}(\alpha\beta\gamma)^2(\alpha\beta\delta)^2(\gamma\delta u)^2\big] \\ = 36(ab\alpha)^2(c\alpha u)^2 + \big[12(a\alpha\beta)^2(b\beta\gamma)^2(\gamma a u)^2 - 6(\alpha\beta\gamma)^2(b\alpha\beta)^2(a\gamma u)^2 \\ - 3(ab\gamma)^2(\alpha\beta\gamma)^2(\alpha\beta u)^2 + 2A(abu)^2\big] + \big[-6(a\tau u)^2 + 6(a\sigma\alpha)^2(u\sigma\alpha)^2 \\ + (a\delta\varepsilon)^2(\delta\alpha\beta)^2(\varepsilon\beta\gamma)^2(a\gamma u)^2 - \tfrac{1}{2}(a\delta\varepsilon)^2(\delta\alpha\beta)^2(\alpha\beta\gamma)^2(\varepsilon\gamma u)^2 \\ + \tfrac{1}{3}A(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{3}(a\delta\varepsilon)^2(\alpha\delta\varepsilon)^2(\alpha\beta\gamma)^2(\beta\gamma u)^2\big] \\ + \big[(\sigma\alpha\beta)^2(\sigma\beta\gamma)^2(\alpha\gamma u)^2 - \tfrac{1}{2}(\sigma\alpha\beta)^2(\alpha\beta\gamma)^2(\sigma\gamma u)^2 + (\tau\alpha\beta)^2(\alpha\beta u)^2\big] \ldots (13)\end{aligned}$$

Again, since Y is of the second order, we may write it $\frac{\phi}{\theta^2}$, where ϕ is an integral function, and $\wp$ is $-\log\theta$. Hence

$$dY = -\frac{2\phi\theta'}{\theta^3} + \frac{\phi'}{\theta^2}, \quad d^2Y - 6Y(d^2\wp) = -4\frac{\phi'\theta' - \phi\theta''}{\theta^3} + \ldots.$$

Also

$$dYd^3\wp - 4Y(d^2\wp)^2 = -2\frac{\phi'\theta' - \phi\theta''}{\theta^3}\frac{\theta'^2}{\theta^2} + \ldots.$$

and therefore

$$dYd^3\wp - 4Y(d^2\wp)^2 - (d^2Y - 6Yd^2\wp),$$

though apparently of the sixth, is really of the fourth order. Hence

$$dYd^3\wp - 4Y(d^2\wp)^2 + (ab\alpha)^2\alpha_x^2(d^2\wp)$$

is of the fourth order. The expression of this in terms of the fourth-order functions may now be computed, and we have the result

$$d\,Y d^3\wp - 4\,Y(d^2\wp)^2 + (aba)^2\alpha_x^2(d^2\wp) = \tfrac{1}{3}(a\alpha\beta)^2(b\beta\gamma)^2$$
$$+ \tfrac{2}{3}(a\alpha\beta)^2(b\alpha\beta)(\alpha\beta\gamma)\,b_x\gamma_x^2 - \tfrac{4}{3}\Delta F - \tfrac{1}{18}(a\alpha\beta)^2(b\alpha\beta)^2F$$
$$+ \tfrac{2}{3}Y(a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{an expression of the third order.} \quad (14)$$

Also, if in (7) we replace u by q, and multiply by q_x^2, we have

$$(p\alpha\beta)^2(p\beta q)\,\alpha_x^2\beta_x p_x^2 + (Lap)\,\alpha_x^3 p_x^2 = 0,$$

or

$$(pq\beta)^2(p\alpha\beta)\,\alpha_x^2\beta_x q_x + (Lp\alpha)\,p_x^2\alpha_x^3 = 0.$$

From this last we readily deduce

$$(Lpu)\,p_x^2 = (pq\alpha)^2(p\alpha u)\,q_x\alpha_x. \quad (15)$$

By means of (8), the right-hand side of (15) may be expressed in terms of functions of the second order, and then (14) and (15) enable us to express any product of a first derivative of Y and a third derivative of $\wp$ as a cubic function of the functions of the second order.

We now obtain two equations from (8) in exactly the same way that (7) and (8) were determined from (5). These two equations are

$$6\,(\xi qu)^3\xi_x = (\eta\alpha\beta)^2(\eta\beta u)^2(\eta\alpha u)\,\alpha_x + (N\alpha u)^3\alpha_x, \quad (16)$$
$$-6\,(\xi\xi' u)^4 = (\theta\alpha\beta)^2(\theta\alpha u)^2(\theta\beta u)^2 + (P\alpha u)^4, \quad (17)$$

where

$$\eta_x^5 = d^5\wp, \quad \theta_x^6 = d^6\wp, \quad P_x^4 = d^4 Y.$$

When (17) is expanded it becomes

$$(abu)^2(cdu)^2 + \tfrac{8}{3}Y(a\alpha u)^2(b\alpha u)^2 + \tfrac{8}{3}(c\alpha\beta)^2(a\alpha u)^2(b\beta u)^2 + \tfrac{5}{36}\Delta(\alpha\beta u)^4$$
$$-\tfrac{1}{3}(abu)^2(c\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{2}{3}(aba)^2(\alpha\beta u)^2(c\beta u)^2 = \ldots. \quad (18)$$

where the terms on the right are at most quadratic in the second order functions.

Now we showed originally that there were 22 relations among the functions of the sixth order that were rational functions of the second derivatives and at most linear in Y. The equation (18) gives 15 of these relations, by equating to zero the various coefficients of powers and products of the u's. There are six others given similarly by equation (13). Also if in (13) we replace u by c, we have a relation independent of those already enumerated, which expresses $Y\Delta$ in terms of second derivatives. We have thus altogether 22. It is almost demonstrable that there are no others, and in fact that all other relations among second and third derivatives, involving Y and its first derivatives, may be derived

by algebraic processes from those given. In particular, there are relations among the second derivatives only. There are apparently none of order so low as 4. They may be obtained by eliminating Y from the 22 relations mentioned. For example, we obtain 15 quintic relations among second derivatives only, by elimination of Y from the equations derived from (13). Similarly we may obtain from (18) and (13) together 201 sextics, though these are not necessarily linearly independent. We notice, however, that the highest-degree terms of all these relations are homogeneous quadratics in the quantities Δ_{pq}, with coefficients quadratic functions of second derivatives. It follows that if we take second derivatives as coordinates in space of six dimensions, all the five-folds mentioned pass doubly through the surface of the eighth order at infinity given by the vanishing of all the first minors of Δ. Now we know *a priori* that these relations must have a common three-fold, and it is clear therefore that this three-fold must be either of the eighth or of the sixteenth order; it seems highly probable that it is of the sixteenth order, and the space at infinity is a trope. This three-fold is the generalization of the Kummer Quartic, which arises when $p=2$, or of the non-singular cubic curve, for $p=1$. We propose to consider it more in detail later.

BRYN MAWR COLLEGE.

Differential Equations Admitting a Given Group.

BY J. EDMUND WRIGHT.

Let there be given a continuous r-parameter group in the n variables $x_1, x_2, \ldots, x_{n-1}, y$, and let the x's be functions of y. Suppose a system of $n-1$ differential equations is given by $I_1 = 0$, $I_2 = 0$, $\ldots$, $I_{n-1} = 0$, where the I's are differential invariants of the group, of orders $a_1, a_2, \ldots, a_{n-1}$ respectively. Then their complete solution involves N arbitrary constants. But since the system is invariant under the given group, it follows that from a solution involving $N-r$ constants we can deduce the general solution, provided the particular solution is not invariant under any subgroup of the given group.

The general solution consists of a set of curves, ∞^N in number, in space of n dimensions, and these curves are merely interchanged under the operations of the group, so that, *e. g.*, from a single curve we may derive by means of the transformations of the group a set of ∞^r curves, provided that the single curve is not invariant under any transformation of the group. All of these are solutions of the differential equations.

Now any curve may be expressed by equations of the type $x_i = f_i(t)$, where f is an appropriate function of t, and t is parametric. If the group be extended by adding the variable t, and assuming $x_1, x_2, \ldots, x_{n-1}, y$ to be functions of t, whilst t is an invariant, we get $n-1$ equations in n dependent variables, all of which are invariants of the group as thus modified.

In cases that frequently arise, *e. g.*, in differential geometry or in dynamics, the independent variable is an invariant of the group, and may itself be taken as t. In other cases we change the independent variable as indicated, and we then need an equation defining t. This equation must be one among the n dependent variables and t, and if it is quite general it may involve derivatives of the dependent variables. Suppose it to be solved for t; then since t is an invariant of the group, if the new system is to remain an invariant system, we

must have a relation of the form $I_n = t$, where I_n is an invariant of the group. For the rest we may choose I_n arbitrarily except that it must be independent of $I_1, I_2, \ldots, I_{n-1}$. We have finally n equations for the n dependent variables as functions of t. Their solution depends on N' constants, where N' may be greater than N, but this gives not only the integral curves of the original system, but also the parametric representation of these integral curves in terms of a particular parameter t. This accounts for the additional constants involved.

Now if I be any invariant of the group, $\frac{dI}{dt}$ is also an invariant. If we regard $\frac{dI}{dt}$ as depending on I, then it may be proved that the group contains precisely n independent invariants in addition to t. We indicate the proof if $r < n$. Suppose that $r-h$ of the operators of zero order are unconnected, $r-h_1$ of the operators of unit order unconnected and so on. Then $h \geq h_1 \geq h_2 \ldots$ and, finally, for some value of λ, h_λ is zero. We have $n-r+h$ invariants of zero order, $2n-r+h_1$ of order ≤ 1, and so on. We thus have $pn-r+h_p$ invariants of order $\leq p$ and $(p-1)n-r+h_{p-1}$ invariants of order $\leq p-1$. Thus there are precisely $n-(h_{p-1}-h_p)$ invariants of order p. Now there are $n-r+h$ invariants of zero order, and therefore this number of invariants of unit order are of the form $\frac{dI}{dt}$. Thus the number of new first-order invariants is

$$n-h+h_1-(n-r+h) = r+h_1-2h.$$

The number of new p-th order invariants is

$$n-(h_{p-1}-h_p)-[n-(h_{p-2}-h_{p-1})] = h_{p-2}+h_p-2h_{p-1}.$$

Thus, finally, the total number of independent invariants is

$$(n-r+h)+(r+h_1-2h)+(h+h_2-2h_1)+\ldots\ldots = n.$$

Let these independent invariants be $X_1, X_2, \ldots, X_n$; then the solution of the differential equations $X_i = F_i(t)$ in the original variables will contain r arbitrary constants. Further, any system of equations of the type $I=0$ may be expressed in terms of $t, X_1, \ldots, X_n, \ldots, \frac{d^i X_k}{dt^i}, \ldots$. Thus the original system may be integrated in two steps. First we have a set of n equations in the n variables X, and t. If these are integrated they give solutions

$$X_1 = F_1(t),\ X_2 = F_2(t),\ \ldots, X_n = F_n(t),$$

which involve N' constants. Now, regarding the X's as functions of the x's, y and their derivatives, we have to integrate a system of invariant equations of order r. But the solutions of this last system are the different curves derived from a single curve by the operations of the group. We therefore need to know a single solution to determine the complete set, provided that the single solution is not invariant under any operation of the group.

For example, consider the group of rotations about the origin in space of three dimensions; the differential operators are

$$y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\; z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},\; x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},$$

and if t is the independent variable the invariants of the group are

$$\Sigma x^2,\; \Sigma x'^2,\; \Sigma x''^2,\; \ldots,\; \Sigma xx',\; \Sigma xx'',\; \ldots,$$

$$\Delta\equiv\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix},\ \text{etc.}$$

We may choose as our invariants X_1, X_2, X_3,

$$X_1=\Sigma x^2,\; X_2=\Sigma x'^2,\; X_3=\Delta,$$

and *then any system of equations invariant under the group will involve for their final solution the determination of a single solution of*

$$X_1=f_1(t),\; X_2=f_2(t),\; X_3=f_3(t).$$

We now take $x\sqrt{f_1(t)}$ instead of x, $y\sqrt{f_1(t)}$ instead of y, and $z\sqrt{f_1(t)}$ instead of z, and introduce instead of t a new variable t_1 defined by

$$dt_1=\sqrt{\frac{f_2(t)}{f_1(t)-\frac{1}{2}f_1'(t)/\sqrt{f_1(t)}}}\,dt.$$

With these changes our system of equations becomes

$$\Sigma x^2=1,\; \Sigma x'^2=1,\; \Delta=f(t),$$

where we now drop the suffix from t_1; and we shall assume that f is not zero. We see that $\Sigma xx'=0$, hence x, y, z, and x', y', z', are the direction cosines of two straight lines at right angles. Also

$$\Sigma xx''=-\Sigma x'^2=-1.$$

Thus

$$\Sigma x(x''+x)=0,\; \Sigma x'(x''+x)=0,$$

and therefore the line perpendicular to the two given lines must have its

direction cosines proportional to $x'' + x$, etc. It follows from the third equation that these direction cosines are precisely $(x'' + x)/f(t)$, etc. From the known relations that exist among the direction cosines of three mutually perpendicular straight lines we have at once

$$\left(\frac{x'' + x}{f}\right)^2 + x'^2 + x^2 = 1, \tag{1}$$

a differential equation of the second order for x. If this be differentiated, it gives the linear equation of the third order

$$x''' - \frac{f'}{f} x'' + (1 + f^2) x' - \frac{f'}{f} x = 0; \tag{2}$$

(it may easily be seen that the apparent solution $x'' + x = 0$ must be excluded). We determine any particular solution of (2) that satisfies (1), and then by elimination we get a first-order equation for y. If we can find any particular solution of the equation for y, z is determinate, and then the complete solution of the set of equations may be at once written down.

As an illustration, suppose that $x = kt$ is a particular solution of (1); then $f = kt/(1 - k^2 - k^2t^2)^{\frac{1}{2}}$. Corresponding particular values of y and z are given by

$$f(1 - k^2)^{\frac{1}{2}} y = kt\,(\cos u + kf \sin u),$$
$$f(1 - k^2)^{\frac{1}{2}} z = kt\,(\sin u - kf \cos u),$$

where $\sin ku = kt/(1 - k^2)^{\frac{1}{2}}$.

If we call these particular values x_0, y_0, z_0, the general solution of the equations is

$$x = l\,x_0 + m\,y_0 + n\,z_0,$$
$$y = l'x_0 + m'y_0 + n'z_0,$$
$$z = l''x_0 + m''y_0 + n''z_0,$$

where the constants l, m, n, etc. are the direction cosines of three mutually perpendicular straight lines.

BRYN MAWR COLLEGE, *November*, 1908.

On the Angles of the Regular Polytopes of Four-dimensional Space.

BY P. H. SCHOUTE.

§ 1. *Notations.*

We introduce general notations for the angles in question, indicating

by a a plane angle between edges (of polygons),
" b " " " faces (of polyhedra),
" c " " " bounding bodies (of polytopes),

these angles varying from naught to $360° = 2\pi$;

by α a solid angle between faces (of polyhedra),
" β " " round an edge (of polytopes),

these angles varying from naught to $720° = 4\pi$;

by A a four-dimensional angle (of polytopes),

this angle varying from naught to $1440° = 8\pi$.

§ 2. *Relations between a, b, c, α, β.*

By means of an isosceles spherical triangle, or one of its two rectangular halves, we can express b in a and c in b by using the cosine-formula, and α in b and β in c by using the expression for the spherical excess. But to that end we ought to know the number p_3 of faces concurring in a vertex for the bounding polyhedron and the number p_4 of limiting polyhedra passing through an edge of the polytope itself. If we add the number p_2 of vertices (and sides) of the limiting polygons of the polytope, we have $a = \pi - \frac{2\pi}{p_2}$; so by the relations we are going to deduce all the angles a, b, c, α, β can be expressed in p_2, p_3, p_4.

Relations (a, b) *and* (α, b).—The three edges of the trihedral angle corresponding to the spherical triangle in view are two adjacent edges of the regular polyhedron and the line joining their common point to the centre of gravity of the polyhedron. So the three angles of the spherical triangle are $\frac{b}{2}$, $\frac{b}{2}$, $\frac{2\pi}{p_3}$, whilst the side opposite to $\frac{2\pi}{p_3}$ is a. So the cosine-formula for the angles gives

$$\cos\frac{2\pi}{p_3} = -\cos^2\frac{b}{2} + \sin^2\frac{b}{2}\cos a,$$

or

$$\sin\tfrac{1}{2}b = \frac{\cos\frac{\pi}{p_3}}{\cos\frac{a}{2}}, \tag{1}$$

and from the spherical excess $b + \frac{2\pi}{p_3} - \pi$ taken p_3 times we deduce

$$\alpha = p_3 b - (p_3 - 2)\pi. \tag{2}$$

Relations (b, c) *and* (β, c).—Here the three edges of the trihedral angle are the lines joining the midpoint of an edge to the centres of gravity of two adjacent faces through that edge and to the centre of gravity of the polytope itself. So the three angles are $\frac{c}{2}$, $\frac{c}{2}$, $\frac{2\pi}{p_4}$; the side opposite to $\frac{2\pi}{p_4}$ is b; so we find

$$\sin\tfrac{1}{2}c = \frac{\cos\frac{\pi}{p_4}}{\cos\frac{b}{2}}, \tag{3}$$

$$\beta = p_4 c - (p_4 - 2)\pi. \tag{4}$$

So, if the number of sides and vertices of the polygons of the polytope is represented by p_2, the formulae (1), (2), (3), (4) enable us indeed to express all the angles a, b, c, α, β in p_2, p_3, p_4.

§ 3. *The Angle A.*

For the first time the angle A was calculated for all the regular polytopes by L. Schläfli in 1852.* We will indicate here in what manner he derived

* See the posthumous work "Theorie der vielfachen Kontinuität" published by Mr. J. H. Graf, p. 118.

his results, though we must omit parts of the demonstration. His chief instrument is a differential formula. Let us consider the simplest case of an angle A formed by four lines l_1, l_2, l_3, l_4 meeting in O and not lying together in the same three-dimensional space. Let us represent by a_{12} the angle between l_1 and l_2, by c_{12} the "adjacent" angle between the spaces $(l_1, l_2)\ l_3$ and $(l_1, l_2)\ l_4$ meeting in the plane l_1, l_2. Then the differential formula under discussion is

$$\delta A = \frac{2}{\pi}\sum_1^6 a\delta c, \tag{I}$$

the unit 180° of A being an eighth part of the whole four-dimensional space round O and the summation Σ extending over the six angles $a_{1,2}, \ldots, a_{3,4}$ and the "adjacent" angles $c_{1,2}, \ldots, c_{3,4}$. Now let us assume any point O' within the angle A and drop perpendiculars from O' to the four limiting spaces of A; then another four-dimensional angle A' is formed, called the *supplementary* four-dimensional angle of A, the angles a' and c' of A' being the supplements of the corresponding angles c and a of A. For this new figure (I) becomes

$$\delta A' = \frac{2}{\pi}\sum_1^6 (\pi - c)\delta(\pi - a) = -\frac{2}{\pi}\sum_1^6 (\pi - c)\delta a. \tag{II}$$

Addition of (I) and (II) gives

$$\delta(A + A') = \frac{2}{\pi}\sum_1^6 \{a\delta c - (\pi - c)\delta a\} = -\frac{2}{\pi}\sum_1^6 \delta\{(\pi - c)a\},$$

which equation can be integrated. As A' becomes 720° when A disappears we get

$$A + A' = 720° - \frac{2}{\pi}\sum_1^6 (\pi - c)a. \tag{III}$$

This formula enables us to find the value of A in the case of the regular polytopes C_5, C_8, C_{120}, the four-dimensional angles of which admit only four edges. But it must be extended in the case of the other regular polytopes to what may be called "regular four-dimensional angles" with more than four edges. A four-dimensional angle is "regular" if there exists a line through its vertex with the property that the space normal to that line in any point P

cuts the edges in the vertices of a regular polyhedron having P for centre. Now it can be shown that the relation (III) holds for any regular four-dimensional angle and the supplementary regular one of the perpendiculars, if the summation Σ be extended to all the equal angles a and the corresponding equal angles c, the number of terms under the sign Σ being therefore equal to the number of edges of the regular polyhedron mentioned above. But if A is a four-dimensional angle of any regular polytope, and the vertex O' of the supplementary angle A' is the centre of that polytope, the edges of A' are the lines joining that centre O to the centres of gravity of the limiting bodies passing through the vertex O of A; so A' is the "central four-dimensional angle" of another regular polytope, viz., of the one polar-reciprocal to the original one. Now, if we designate—in accordance with my German text-book—by e, k, f, r the numbers of vertices, edges, faces, limiting bodies of the original regular polytope, these quantities taken in reversed order r, f, k, e represent at the same time the numbers of vertices, edges, faces, limiting bodies of the polar-reciprocal one. We find, therefore,

$$A' = \frac{1440^\circ}{e}.$$

Moreover, the number of terms under the sign Σ can be expressed in p_2, e, f. For $p_2 f$ represents how many times any vertex lies in any face, and therefore at the same time how many times any face passes through any vertex; so $\frac{p_2 f}{e}$ represents the number of faces passing through the vertex of A, *i. e.*, the number of edges of the regular polyhedron. So by introducing $\pi - \frac{2\pi}{p_2}$ for a we find, after slight reductions, the general result,*

$$A = \frac{1}{e}\left\{720^\circ(e-2) - 2f(p_2-2)(\pi - c)\right\}. \tag{5}$$

§ 4. *The General Results for Regular Polytopes.*

By applying the general formula to the six different regular polytopes we get the following table of results, where s stands for $\sqrt{5}$.

* Probably this formula giving at once the angle A for *all* the regular polytopes is new.

	p_2	p_3	p_4	a	b	c	α	β	A
C_5	3	3	3	60° $\frac{1}{2}$	70° 31′ 44″ $\frac{1}{3}$	75° 31′ 21″ $\frac{1}{4}$	31° 35′ 11″ $\frac{23}{27}$	46° 34′ 3″ $\frac{11}{16}$	14° 5′ 24″ $= 4c - \frac{8}{5}\pi$ $\cos 4c = \frac{17}{32}$
C_8	4	3	3	90° 0	90° 0	90° 0	90° 0	90° 0	90° 0
C_{16}	3	3	4	60° $\frac{1}{2}$	70° 31′ 44″ $\frac{1}{3}$	120° $-\frac{1}{2}$	31° 35′ 11″ $\frac{23}{27}$	120° $-\frac{1}{2}$	60° $\frac{1}{2}$
C_{24}	3	4	3	60° $\frac{1}{2}$	109° 28′ 16″ $-\frac{1}{3}$	120° $-\frac{1}{2}$	77° 53′ 6″ $\frac{17}{81}$	180° -1	180° -1
C_{120}	5	3	3	108° $-\frac{1}{4}(e-1)$	116° 33′ 54″ $-\frac{1}{e}$	144° $-\frac{1}{4}(e+1)$	169° 41′ 48″ $-\frac{11}{5e}$	252° $\frac{1}{8}(3+e)$	458° 24′
C_{600}	3	3	5	60 $\frac{1}{2}$	70° 31′ 44″ $\frac{1}{3}$	164° 28′ 39″ $-\frac{1}{8}(3e+1)$	31° 35′ 11″ $\frac{23}{27}$	282° 23′ 15″ $\frac{1}{128}(61-15e)$	397° 33′ $= \frac{58}{5}\pi - 20c$ $\cos 20c = \frac{701777}{2097152}$

This table gives the angles in degrees and besides the cosine of every angle, with exception of the case A_{120} where the angle *is* mensurable and the cosine *is not.*

§ 5. *The Angle A Continued. Conclusion.*

We indicate here how the angle A of the less complicate cells, C_5, C_8, C_{16}, C_{24}, can be found without aid of Schläfli's differential formula.

The Case C_5.—By a regular truncation of C_5 at the vertices half-way up the edges, *i. e.*, such that the truncating spaces pass through the mid-points of the edges, we obtain* a polytope (10, 30, 30, 10) with ten vertices of the same kind limited by five octahedra and five tetrahedra. If we distinguish the four-dimensional angle of this new polytope from that angle A of C_5 as A', the two equations

$$2A + A' = 2(3c - \pi), \quad A + 2A' = \frac{4}{5}\pi$$

give $A = 4c - \frac{8}{5}\pi$ by elimination of A'. So we have only to prove these two equations.

The first of the two equations is found by remarking that the portion of four-dimensional space round the midpoint of an edge of C_5 inclosed by the three

* See *Proceedings*, Amsterdam, Vol. X, p. 499.

limiting spaces of C_5 passing through that edge is divided by the truncation into A' and the four-dimensional angles at that point of the two regular five-cells cut off at either side, and that its ratio to 1440° is equal to that of its angle β to 720°. So we find

$$2A + A' = 2\beta = 2(3c - \pi).$$

The second of the two equations has quite another origin. If we cut the net of five-dimensional measure-polytopes by a four-dimensional space passing through a vertex normal to a diagonal, we obtain* a four-dimensional space-filling consisting of regular five-cells and of polytopes (10, 30, 30, 10) mentioned above. The 2^5 measure-polytopes round the chosen point project themselves on the diagonal in groups of

$$(1, 5, 10, 10, 5, 1);$$

of these the ten polytopes of the two groups of five are cut in five-cells, the twenty polytopes of the two groups of ten in polytopes (10, 30, 30, 10), whilst the two polytopes of the two extreme groups are left undivided. So we find that round the chosen point the four-dimensional space is filled by $10A$ and $20A'$ or

$$10A + 20A' = 1440°.$$

The values found for c_5, β_5, A_5 furnish us limits for the numbers of cells C_5 that can be arranged in four-dimensional space round a face, round an edge, round a vertex successively; these limits are evidently the integers contained in $\frac{360°}{c_5}$, $\frac{720°}{\beta_5}$, $\frac{1440°}{A_5}$, *i. e.*, 4, 15, 102. But probably the limits 15, 102 are still too large. For instance, in the analogous question about the tetrahedron in three-dimensional space $\frac{720°}{a_5}$ gives the limit 22, whilst the icosahedron teaches us that we can only be sure of this, that 20 tetrahedra can concur in a common vertex, the central solid angle of the icosahedron being $36° > a_5$, whilst it will probably be impossible to arrange the 20 icosahedra round a point in such a way as to leave place for a new one.

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*See *Proceedings*, Amsterdam, Vol. X, p. 688.

The Case C_8.—From the selfevident space-filling by four-dimensional measure-polytopes we deduce immediately

$$16A_8 = 1440^\circ, \text{ i. e., } A_8 = 90^\circ.$$

Here we find that it *is* possible to arrange $4C_8$ round a face, $8C_8$ round an edge, $16C_8$ round a point.

The Case C_{16}.—A C_{16} is generated by taking away from a C_8 rectangular five-cells by a regular truncation at the eight vertices of one of the two octuples of non-adjacent vertices extended so far as to remove all the original edges.* As four of the removed five-cells meet in any remaining vertex and each acute four-dimensional angle of these five-cells is an eighth part of A_{16}, the five-cell itself being a sixteenth part of C_{16}, we find

$$A_{16} = A_8 - \frac{4}{8} A_{16}, \text{ i. e., } A_{16} = 60^\circ.$$

This angle A is the smallest of the whole lot.

The same result is deduced from the well-known four-dimensional space-filling by C_{16}. By transformation of the net (C_8) into a net (C_{16}) we find indeed that it *is* possible to arrange $3C_{16}$ round a face, $6C_{16}$ round an edge, $24C_{16}$ round a point.

The Case C_{24}.—A C_{24} is obtained by joining to a C_8 at each of the eight limiting cubes a regular four-dimensional pyramid, the base of which is that cube, whilst the height is half the edge of the cube. So we get

$$A_{24} = A_8 + \frac{4}{4} A_8 = 180^\circ,$$

in accordance with the well-known four-dimensional space-filling by cells C_{24}, from which we deduce that it *is* possible to arrange $3C_{24}$ round a face, $4C_{24}$ round an edge, $8C_{24}$ round a point.

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* See my "Mehrdimensionale Geometrie", Vol II, p. 242, and *Proceedings*, Amsterdam, Vol. X, p. 536–545.

If we indicate the numbers of cells round a face, round an edge, round a point by q_2, q_1, q_0 successively, we have the results laid down in the following small table

	q_2	q_1	q_0
C_5	4	15 ?	102 ?
C_8	**4**	**8**	**16**
C_{16}	**3**	**6**	**24**
C_{24}	**3**	**4**	**8**
C_{120}	2	2	3
C_{600}	2	2	3

The figures in heavy type correspond to the cases of space-filling.

The Asymptotic Representation of the Elliptic Cylinder Functions.

By William Marshall.

Introduction.

We are indebted to E. Heine* for the first serious treatment of the functions of the elliptic cylinder. Starting from the defining equation

$$\frac{d^2E}{d\phi^2}+\left(\frac{8}{b}\cos 2\phi+4z\right)E=0,$$

where b and z are constants, Heine shows that $E(\phi)$ can be expressed in the form of the following infinite series:

$$E(\phi)=\tfrac{1}{2}a_0+a_1\cos 2\phi+a_2\cos 4\phi+\ldots,$$

where the coefficients, functions of b and z, are subject to a certain recurrence formula, namely,

$$a_{n+1}=b(n^2-z)a_n-a_{n-1}.$$

But this series, as was pointed out by Heine, is not convergent for all values of b and z, but only for certain *particular* values which he found as the roots of a certain function of b and z. This function, however, was not carefully defined by Heine, nor was his proof of the convergence of the series sufficiently rigorous to satisfy modern mathematical requirements.

S. Dannacher† cleared up the inaccuracies and supplied the deficiencies in Heine's presentation. He subjected the function whose roots determine the convergence of the series to a careful investigation, and showed how these roots might be calculated. He gave also a satisfactory proof of the convergence of the series.

* E. Heine, Kugelfunktionen, Bd. I, § 104.

† Inaugural Dissertation, Zürich, 1906.

W. H. Butts* made a particular study of the roots of this so-called *limiting function*, and calculated for different values of b a considerable number numerically.

In the present paper, following an idea originally due to Stokes,† we obtain asymptotic or semi-convergent developments for the elliptic cylinder functions. These asymptotic expansions, which hold approximately for reasonably large values of the argument, have two decided advantages. In the first place these series are extremely well adapted to the calculation of the numerical values of the function when the argument is large, or even reasonably large; and in the second place they indicate without serious calculation the behavior of the function for large values of the argument, and particularly for an infinite argument, and they afford the most satisfactory method of determining where the function vanishes.

I. *The Transformation of the Equation.*

The equation of the functions of the elliptic cylinder may be written in the form

$$\frac{d^2U}{du^2} + (k^2\cosh^2 u + \mathfrak{B})U = 0, \tag{1}$$

where k and $\mathfrak{B}$ are constants. If we change the independent variable by putting $e^u = z$,‡ since

$$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

the equation (1) goes over into

$$\frac{d^2U}{dz^2} + \frac{1}{z}\cdot\frac{dU}{dz} + \left[\frac{k^2}{4}\left(1 + \frac{2}{z^2} + \frac{1}{z^4}\right) + \frac{\mathfrak{B}}{z^2}\right]U = 0. \tag{2}$$

We can remove the term containing the first derivative by putting $U_1 = Uz^{\frac{1}{2}}$. Then (2) assumes the form

$$\frac{d^2U_1}{dz^2} + \left[\frac{k^2}{4} + \left(\frac{k^2}{2} + \frac{1}{4} + \mathfrak{B}\right)\frac{1}{z^2} + \frac{k^2}{4z^4}\right]U_1 = 0. \tag{3}$$

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* Inaugural Dissertation, Zürich, 1908.

† Trans. Cambridge Phil. Soc., Vol. IX, Part I, or Math. and Phys. Papers of G. G. Stokes, Vol. II, p. 329.

‡ As a result of this transformation, or, in general, $e^{au} = z$, the singular points of the resulting equation lie only at 0 and ∞. Of this family of transformations, $a = 1$ is the only one which reduces the equation to a form such that, for large values of z, it becomes approximately (9); and this form is desirable for our present purpose.

Finally, if we put $\frac{kz}{2} = x$, (3) becomes

$$\frac{d^2U_1}{dx^2} + \left[1 + \left(\frac{k^2}{2} + \frac{1}{4} + \mathfrak{B}\right)\frac{1}{x^2} + \frac{k^4}{16}\cdot\frac{1}{x^4}\right] U_1 = 0, \tag{4}$$

or, as we may write,

$$\frac{d^2U_1}{dx^2} + \left(1 + \frac{p}{x^2} + \frac{q}{x^4}\right) U_1 = 0, \tag{5}$$

where the following relations exist between the quantities involved in (5) and in (1):

$$\left.\begin{aligned} p &= \frac{k^2}{2} + \frac{1}{4} + \mathfrak{B}, \\ q &= \frac{k^4}{16}, \\ U_1 &= Ue^{\frac{u}{2}}, \\ x &= \frac{ke^u}{2}. \end{aligned}\right\} \tag{6}$$

II. *The Solutions in the Neighborhood of the Singularities.*

The point $x = 0$ is an essential singular point of the equation (5), as is also the point $x = \infty$. For if we put $x = \frac{1}{\xi}$, (5) becomes

$$\frac{d^2U_1}{d\xi^2} + \frac{2}{\xi}\cdot\frac{dU_1}{d\xi} + \left(\frac{1}{\xi^4} + \frac{p}{\xi^2} + q\right) U_1 = 0, \tag{7}$$

so that $\xi = 0$ is an essential singularity of (7).

Following the known method, we might now obtain an asymptotic solution of (5) in the neighborhood of $x = 0$ by putting

$$U_1 = e^{\frac{\lambda}{x}} x^{\alpha}(a_0 + a_1 x + a_2 x^2 + \dots\dots), \tag{8}$$

and then determining α, λ and the coefficients $a_0, a_1, a_2, \dots\dots$ so that (5) is formally satisfied; it will, however, in this case, be somewhat simpler first to obtain a solution in the neighborhood of $x = \infty$, and from this to obtain the solution which holds in the neighborhood of $x = 0$.

For large values of x, (5) becomes approximately

$$\frac{d^2 U_1}{dx^2} + U_1 = 0, \tag{9}$$

of which the complete solution is

$$U_1 = C_1 \cos x + C_2 \sin x. \tag{10}$$

Our solution of (5), then, which is to hold in the neighborhood of $x = \infty$ must be such that for large values of x it assumes approximately the form (10). We assume, then, as an asymptotic solution in the neighborhood of $x = \infty$,

$$U_1 = \sin x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots\right) + \cos x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right), \tag{11}$$

where the $A_0, A_1, A_2, \ldots, B_0, B_1, B_2, \ldots$ are constants which must be determined so that the equation is formally satisfied. We have, then, from (11):

$$U_1' = \sin x\left(-\frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots\right) + \cos x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots\right)$$
$$+ \cos x\left(-\frac{B_1}{x^2} - \frac{2B_2}{x^3} - \frac{3B_3}{x^4} - \ldots\right) - \sin x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right),$$

$$U_1'' = \sin x\left(\frac{1.2A_1}{x^3} + \frac{2.3A_2}{x^4} + \frac{3.4A_3}{x^5} + \ldots\right) + 2\cos x\left(-\frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots\right)$$
$$- \sin x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots\right) + \cos x\left(\frac{1.2B_1}{x^3} + \frac{2.3B_2}{x^4} + \frac{3.4B_3}{x^5} + \ldots\right)$$
$$- 2\sin x\left(-\frac{B_1}{x^2} - \frac{2B_2}{x^3} - \frac{3B_3}{x^4} - \ldots\right) - \cos x\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right).$$

When we now substitute this and the value of U_1 from (11) in equation (5) and arrange according to descending powers of x, we have for the left-hand side:

$$\sin x \left\{ \begin{array}{rrrrrr} & \frac{1.2A_1}{x^3} + & \frac{2.3A_2}{x^4} + & \frac{3.4A_3}{x^5} + & \frac{4.5A_4}{x^6} + & \ldots \\ + \frac{2.1B_1}{x^2} + & \frac{2.2B_2}{x^3} + & \frac{2.3B_3}{x^4} + & \frac{2.4B_4}{x^5} + & \frac{2.5B_5}{x^6} + & \ldots \\ + \frac{pA_0}{x^2} + & \frac{pA_1}{x^3} + & \frac{pA_2}{x^4} + & \frac{pA_3}{x^5} + & \frac{pA_4}{x^6} + & \ldots \\ & & + \frac{qA_0}{x^4} + & \frac{qA_1}{x^5} + & \frac{qA_2}{x^6} + & \ldots \end{array} \right.$$

$$+\cos x\left\{\begin{array}{l} \qquad\qquad\qquad \dfrac{1.2B_1}{x^3}+\dfrac{2.3B_2}{x^4}+\dfrac{3.4B_3}{x^5}+\dfrac{4.5B_4}{x^6}+\dots\\ -\dfrac{2.1A_1}{x^2}-\dfrac{2.2A_2}{x^3}-\dfrac{2.3A_3}{x^4}-\dfrac{2.4A_4}{x^5}-\dfrac{2.5A_5}{x^6}-\dots\\ +\dfrac{pB_0}{x^2}+\dfrac{pB_1}{x^3}+\dfrac{pB_2}{x^4}+\dfrac{pB_3}{x^5}+\dfrac{pB_4}{x^6}+\dots\\ \qquad\qquad\qquad +\dfrac{qB_0}{x^4}+\dfrac{qB_1}{x^5}+\dfrac{qB_2}{x^6}+\dots \end{array}\right.$$

In order that this vanish, the coefficients of $\cos x$ and $\sin x$ must vanish separately. The necessary conditions for this are that the coefficients of the various powers of x, namely, x^{-2}, x^{-3}, x^{-4}, ..., should vanish separately. This gives us the following relations for determining the A's and B's in terms of any two of them, say in terms of A_0 and B_0:

$$\left.\begin{array}{l}
A_1=\dfrac{p}{2}B_0=B_0f_1(p,q)\ \text{(say)},\\
B_1=-\dfrac{p}{2}A_0=-A_0f_1,\\
A_2=\dfrac{B_1(p+2)}{2.2}=-\dfrac{A_0}{2^2.2}p(p+2)=-A_0f_2,\\
B_2=-\dfrac{A_1(p+2)}{2.2}=-\dfrac{B_0}{2^2.2}p(p+2)=-B_0f_2,\\
A_3=B_0\left[\dfrac{-p(p+2)(p+6)}{2^3.3\,!}+\dfrac{q}{2.3}\right]=-B_0f_3,\\
B_3=A_0\left[\dfrac{p(p+2)(p+6)}{2^3.3\,!}-\dfrac{q}{2.3}\right]=A_0f_3,\\
A_4=A_0\left[\dfrac{p(p+2)(p+6)(p+12)}{2^4.4\,!}-\dfrac{q(p+12)}{2.4.2.3}-\dfrac{pq}{2.1.2.4}\right]=A_0f_4,\\
B_4=B_0\left[\dfrac{p(p+2)(p+6)(p+12)}{2^4.4\,!}-\dfrac{q(p+12)}{2.4.2.3}-\dfrac{pq}{2.1.2.4}\right]=B_0f_4,\\
A_5=B_0\left[\dfrac{p(p+2)(p+6)(p+12)(p+20)}{2^5.5\,!}-\dfrac{(p+12)(p+20)}{2.4.2.3.2.5}\right.\\
\qquad\qquad\qquad\left.-\dfrac{pq(p+20)}{2.1.2.4.2.5}-\dfrac{pq(p+2)}{2^5.2^2.2}\right]=B_0f_5,\\
B_5=-A_0f_5,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}\right\}\quad(12)$$

Here $f_1, f_2, f_3, \ldots.$ are simply abbreviations. We have, in general,

$$\left.\begin{aligned} A_n &= \frac{B_{n-1}[p+n(n-1)]}{2(n-1)} + \frac{B_{n-3}q}{2(n-1)}, \\ B_n &= -\frac{A_{n-1}[p+n(n-1)]}{2(n-1)} - \frac{A_{n-3}q}{2(n-1)}. \end{aligned}\right\} \tag{13}$$

From these relations we may determine the constants $A_1, A_2, A_3, \ldots.,$ $B_1, B_2, B_3, \ldots.$ as far as we choose, though, as might have been expected, it is not possible to give general formulas for A_n and B_n except in terms of the coefficients which immediately precede them. Substituting now in (11) the values of the coefficients as given by (12), we thus obtain for the asymptotic solution of (5) for large values of x the following:

$$\begin{aligned} U_1 = \sin x\left[A_0 + \frac{B_0 f_1}{x} - \frac{A_0 f_2}{x^2} - \frac{B_0 f_3}{x^3} + \frac{A_0 f_4}{x^4} + \ldots.\right] \\ + \cos x\left[B_0 - \frac{A_0 f_1}{x} - \frac{B_0 f_2}{x^2} + \frac{A_0 f_3}{x^3} + \frac{B_0 f_4}{x^4} - \ldots.\right]. \end{aligned} \tag{14}$$

This we may write in the form

$$\begin{aligned} U_1 = A_0\left(\sin x - \frac{f_1\cos x}{x} - \frac{f_2 \sin x}{x^2} + \frac{f_3\cos x}{x^3} + \ldots.\right) \\ + B_0\left(\cos x + \frac{f_1 \sin x}{x} - \frac{f_2\cos x}{x^2} - \frac{f_3\sin x}{x^3} + \ldots.\right). \end{aligned} \tag{15}$$

If now we change the arbitrary constants by putting

$$\left.\begin{aligned} A_0 \sin x + B_0 \cos x &= C\cos(\alpha - x), \\ -(A_0\cos x - B_0 \sin x) &= C \sin(\alpha - x), \end{aligned}\right\} \tag{16}^*$$

we may write (14) or (15) in the form

$$\begin{aligned} U_1 = C\cos(\alpha - x)\left[1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \ldots.\right] \\ + C\sin(\alpha - x)\left[\frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \ldots.\right]; \end{aligned} \tag{17}$$

or we may write

$$U_1 = C[P\cos(\alpha - x) + Q\sin(\alpha - x)], \tag{18}$$

* This amounts to putting

$$C = \sqrt{A_0^2 + B_0^2},$$
$$\alpha = \tan^{-1}\frac{A_0}{B_0}.$$

where P and Q denote the following infinite series:

$$\left.\begin{aligned} P &= 1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \dots, \\ Q &= \frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \dots \end{aligned}\right\} \tag{19}$$

If we return to the original variables U and u, we may write (18) in the form (see the relations (6))

$$U = \frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right], \tag{20}$$

where, in P and Q, x is replaced by $\frac{ke^u}{2}$ and p and q by their equivalents.

The series denoted by P and Q are at first, for reasonably large values of x, rapidly convergent; they are both, however, ultimately rapidly divergent, as may be seen from the relations (12). The point at which they begin to diverge depends on the relative magnitude of f_n and x^n. In fact if we stop after taking $2n$ terms of the solution (14) (n terms containing $\sin x$ and n containing $\cos x$) and set the result in (5), we obtain

$$\begin{aligned} \frac{d^2U_1}{dx^2} + \left(1 + \frac{p}{x^3} + \frac{q}{x^4}\right)U_1 = \frac{f_{n+1}}{x^{n+1}}\Big[A_0 \sin\left(x + \frac{n-1}{2}\pi\right) \\ + B_0\cos\left(x + \frac{n-1}{2}\pi\right)\Big]; \end{aligned} \tag{21}$$

and the right-hand side of this is small if $\frac{f_{n+1}}{x^{n+1}}$ is also small. In using such series in numerical computation, we need not take all the terms up to and including the smallest; the number taken will depend upon the closeness of the approximation desired.

To obtain a solution of equation (5) which holds in the neighborhood of the other singularity $x = 0$, we proceed most conveniently as follows: in (5) we put $x = \frac{1}{t}$; (5) becomes then

$$\frac{d^2U_1}{dt^2} + \frac{2}{t}\frac{dU_1}{dt} + \left(\frac{1}{t^4} + \frac{p}{t^3} + q\right)U_1 = 0. \tag{22}$$

We can get rid of the term containing the first derivative by putting $U_1 = U_2 t^{-1}$. Then (22) becomes

$$U_2'' + \left(\frac{1}{t^4} + \frac{p}{t^3} + q\right)U_2 = 0. \tag{23}$$

Finally, if we put $t=\xi q^{-\frac{1}{2}}$, we obtain

$$\frac{d^2 U_2}{d\xi^2}+\left(1+\frac{p}{\xi^2}+\frac{q}{\xi^4}\right)U_2=0. \tag{24}$$

Equation (24) is, however, of exactly the same form as (5), so that we may use the former solution (14) or (15). The solution of (24) which is valid in the neighborhood of $\xi=\infty$ is then

$$U_2=\sin\xi\left(a_0+\frac{b_0 f_1}{\xi}-\frac{a_0 f_2}{\xi^2}-\frac{b_0 f_3}{\xi^3}+\ldots\right.$$
$$\left.+\cos\xi\left(b_0-\frac{a_0 f_1}{\xi}-\frac{b_0 f_2}{\xi^2}+\frac{a_0 f_3}{\xi^3}+\ldots\right.,\right. \tag{25}$$

where the f_1, f_2, f_3, ... have exactly the same meaning as in (12). If now in (25) we restore the values of the variables U_2 and ξ, we have

$$U_1=x\sin\frac{q^{\frac{1}{2}}}{x}\left(a_0+\frac{x\,b_0 f_1}{q^{\frac{1}{2}}}-\frac{x^2 a_0 f_2}{q}-\frac{x^3 b_0 f_3}{q^{\frac{3}{2}}}+\ldots\right.$$
$$\left.+x\cos\frac{q^{\frac{1}{2}}}{x}\left(b_0-\frac{x\,a_0 f_1}{q^{\frac{1}{2}}}-\frac{x^2 b_0 f_2}{q}+\frac{x^3 a_0 f_3}{q^{\frac{3}{2}}}+\ldots\right.\right. \tag{26}$$

We may write (26) in the form

$$\begin{aligned}U_1=a_0 x&\left\{\sin\frac{q^{\frac{1}{2}}}{x}\left(1-\frac{x^2}{q}f_2+\frac{x^4}{q^2}f_4-\ldots\right)\right.\\&\left.+\cos\frac{q^{\frac{1}{2}}}{x}\left(-\frac{x}{q^{\frac{1}{2}}}f_1+\frac{x^3}{q^{\frac{3}{2}}}f_3-\frac{x^5}{q^{\frac{5}{2}}}f_5+\ldots\right)\right\}\\+b_0 x&\left\{\sin\frac{q^{\frac{1}{2}}}{x}\left(\frac{x}{q^{\frac{1}{2}}}f_1-\frac{x^3}{q^{\frac{3}{2}}}f_3+\frac{x^5}{q^{\frac{5}{2}}}f_5-\ldots\right)\right.\\&\left.+\cos\frac{q^{\frac{1}{2}}}{x}\left(1-\frac{x^2}{q}f_2+\frac{x^4}{q^2}f_4-\ldots\right)\right\}.\end{aligned} \tag{27}$$

We may adopt then, as the two fundamental integrals of equation (5) which are valid asymptotically in the neighborhood of $x=0$:

$$U_{1p}=x\left(A\sin\frac{q^{\frac{1}{2}}}{x}-B\cos\frac{q^{\frac{1}{2}}}{x}\right), \tag{28}$$

$$U_{1q}=x\left(B\sin\frac{q^{\frac{1}{2}}}{x}+A\cos\frac{q^{\frac{1}{2}}}{x}\right), \tag{29}$$

where

$$\left.\begin{aligned}A&=1-\frac{x^2}{q}f_2+\frac{x^4}{q^2}f_4-\frac{x^6}{q^3}f_6+\ldots,\\B&=\frac{x}{q^{\frac{1}{2}}}f_1-\frac{x^3}{q^{\frac{3}{2}}}f_3+\frac{x^5}{q^{\frac{5}{2}}}f_5-\ldots\end{aligned}\right\} \tag{30}$$

These integrals have no meaning when $q = 0$. This case, however, we need not consider, since for $q = 0$ (5) becomes

$$U_1'' + \left(1 + \frac{p}{x^2}\right) U_1 = 0, \qquad (31)$$

and this is a reduced Bessel equation with $n^2 = \frac{1}{4} - p$.*

It remains to bring the integrals (18) and (28) or (29) into connection; that is, to determine the constants C and α in (18) so that (18) shall be an approximation to the fundamental integrals which are denoted by U_{1p} and U_{1q}' in (28) and (29). In general this is brought about by expressing the function, here U_{1p} or U_{1q}, in the form of a definite integral and then, from this, determining the leading term in the function as the argument increases. But the expressing of these functions U_{1p} and U_{1q} in the form of definite integrals of sufficient simplicity seems to involve difficulties not as yet overcome.† However, for any particular case, for any particular pair of values of p and q, the constants C and α might be determind in the following manner: Having chosen a value of x for which (18) and (28), for example, give a sufficiently close approximation, compute the numerical values of these expressions (18) and (28) for this value of x. This would give one relation between C and α. By proceeding in a similar way for a second value of x, the values of C and α would be determined.‡

III. *The Roots of* $U_\alpha = 0$.

Although we have not determined the constants C and α in (18) so that (18) shall be an approximation to the fundamental integrals U_{1p} and U_{1q}, yet it is possible to determine from (18) the general behavior of these functions for large values of x. For the sake of simplicity we carry the discussion through with the variables U_1 and x and denote by $U_{1\alpha}$ either of the integrals U_{1p} or U_{1q}, where the α will serve to indicate that on the value of the constants α and C

* That is, if in (31) we put $U_1 = Jx^{\frac{1}{2}}$, we have, after reduction,

$$J'' + \frac{J'}{x} + J\left(1 + \frac{p - \frac{1}{4}}{x^2}\right) = 0;$$

and this is the ordinary form of the Bessel equation with n^2 replaced by $\frac{1}{4} - p$.

† See III, where the value of α is approximately determined, but not in this general manner.

‡ See the numerical example in V. Here the arbitrary constant is determined in the above-mentioned way.

will depend whether the one or the other of these integrals is intended. We have then, from (18),

$$U_{1\alpha} = C[P \cos(\alpha - x) + Q \sin(\alpha - x)]; \tag{31}$$

and we proceed to find where $U_{1\alpha}$ vanishes. We put

$$P = M \cos\psi, \quad Q = M \sin\psi, \tag{32}$$

where, as before,

$$\left.\begin{aligned} P &= 1 - \frac{f_2}{x^2} + \frac{f_4}{x^4} - \frac{f_6}{x^6} + \ldots, \\ Q &= \frac{f_1}{x} - \frac{f_3}{x^3} + \frac{f_5}{x^5} - \frac{f_7}{x^7} + \ldots \end{aligned}\right\} \tag{19}$$

(31) becomes then

$$U_{1\alpha} = CM[\cos(\alpha - x)\cos\psi + \sin(\alpha - x)\sin\psi], \tag{33}$$

$$U_{1\alpha} = CM \cos(\psi - \alpha + x), \tag{34}$$

where, from (32), since

$$\left.\begin{aligned} M &= \sqrt{P^2 + Q^2}, \\ \psi &= \tan^{-1}\frac{Q}{P}, \end{aligned}\right\} \tag{35}$$

$$\psi = \tan^{-1}\left(\frac{f_1}{x} + \frac{f_1 f_2 - f_3}{x^3} + \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3}{x^5} + \ldots\right), \tag{36}$$

$$M = 1 - \frac{f_2 - \frac{f_1^2}{2}}{x^2} + \frac{f_4 - f_1 f_3 + \frac{f_1^2 f_2}{2} - \frac{f_1^4}{8}}{x^4} - \ldots \tag{37}$$

Now, from (34), $U_{1\alpha}$ will be zero when

$$\cos(\psi - \alpha + x) = 0;$$

that is, when

$$\psi - \alpha + x = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \ldots, \ \frac{2n-1}{2}\cdot\pi;$$

that is, when

$$x = \alpha + \frac{2n-1}{2}\cdot\pi - \psi \tag{38}$$

(where n is an integer). This equation we have to solve for x on the supposition that x is large. If in (38) we put, for shortness, $\alpha + \frac{2n-1}{2}\cdot\pi = \phi$, then we have to solve

$$x = \phi - \psi. \tag{39}$$

If now we expand ψ in terms of x by means of the expansion of $\tan^{-1} z$, namely

$$\tan^{-1} z = z - \frac{1}{3} z^3 + \frac{1}{5} z^5 - \frac{1}{7} z^7 + \ldots,$$

we obtain, after reduction,

$$\psi = \frac{f_1}{x} + \frac{f_1 f_2 - f_3 - \frac{f_1^3}{3}}{x^3} + \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3 - f_1^3 f_2 + f_1^2 f_2 + \frac{f_1^5}{5}}{x^5} + \ldots \quad (40)$$

Putting this value of ψ in (39) and solving the resulting equation, namely

$$x = \phi - \frac{f_1}{x} - \frac{f_1 f_2 - f_3 - \frac{f_1^3}{3}}{x^3} - \frac{f_5 - f_1 f_4 - f_1 f_2^2 + f_2 f_3 - f_1^3 f_2 + f_1^2 f_3 + \frac{f_1^5}{5}}{x^5} + \ldots, \quad (41)$$

by the method of successive approximations, we obtain

$$\begin{aligned} x = \phi - f_1 \cdot \frac{1}{\phi} + \left(f_3 - f_1 f_2 + \frac{f_1^3}{3} - f_1^2\right) \cdot \frac{1}{\phi^3} \\ - \Big(f_5 - f_1 f_4 - f_2 f_3 + f_1 f_2^2 + f_1^2 f_3 - f_1^3 + \frac{f_1^5}{5} \\ + 4 f_1 f_3 - 4 f_1^2 f_2 + \frac{f_1^4}{3} + 2 f_1^3\Big) \frac{1}{\phi^5} + \ldots \end{aligned} \quad (42)$$

We have, then, for the general formula for the nth root of the equation $U_{1a} = 0$, the following:

$$x = \alpha + \frac{2n-1}{2}\pi - \frac{f_1}{\alpha + \frac{2n-1}{2}\pi} + \frac{f_3 - f_1 f_2 + \frac{f_1^3}{3} + f_1^2}{\left(\alpha + \frac{2n-1}{2}\pi\right)^3} - \ldots \quad (43)$$

By means of the relations (12) we may express this in terms of p and q. We find, after some reduction,

$$x = \alpha + \frac{2n-1}{2}\pi - \frac{p}{2\alpha + (2n-1)\pi} + \frac{5p^2 + 6p - 12q}{3[2\alpha + (2n-1)\pi]^3} - \ldots \quad (44)$$

By means of the relations (6) this may be transformed into terms of the original quantities U, u, k and $\mathfrak{B}$.

The expression (44) shows clearly how the roots are spaced along the axis. It shows also distinctly the influence of the undetermined constant on the position of the roots. The leading term in (44) is of course $\alpha + \frac{2n-1}{2}\pi$, so that a change in the value of α amounts to a displacement of the roots along the axis. (44) shows also how the position of the roots depends upon the values of p and q. In (44) q appears first in the fourth term, which for ordinary values of p and q, on account of the rapid convergence of the series, will be small, so that a change in the value of q will produce a very slight change in the position of the roots.

In the calculations involved in the preceding pages — that is, in finding the values of ψ as given in (36) and (40), and M as given in (37), and in solving the equation (41) — we may proceed as there indicated, performing the arithmetical operations upon the infinite series there involved, and carrying the results to the desired degree of accuracy. These results may be obtained more briefly (particularly if many terms are desired) and much more elegantly by the following method.*

We had found, as a solution of (5),

$$U_{1a} = C[P\cos(\alpha - x) + Q\sin(\alpha - x)]. \tag{31}$$

If now we put

$$P = M\cos\psi, \quad Q = M\sin\psi, \tag{32}$$

and further set $C = 1$, $\alpha = 0$, we have, as a particular solution of (5),

$$U_1 = M\cos(\psi + x). \tag{45}$$

By substituting this integral in the differential equation (5), and determining the coefficients so that (5) is satisfied, we should be able to deduce recurrence formulas for the coefficients of M and ψ. We have then, from (45),

$$U_1' = M'\cos(x + \psi) - M\sin(x + \psi)(1 + \psi'), \tag{46}$$

$$U_1'' = M''\cos(x + \psi) - 2M'\sin(x + \psi)(1 + \psi') \\ - M\sin(x + \psi)\psi'' - M\cos(x + \psi)(1 + \psi')^2. \tag{47}$$

* This method was suggested to me by Professor Burkhardt, who, however, has not published anything on the subject. It is generally applicable in similar problems, being particularly advantageous in the calculation of the roots of Bessel functions.

When we substitute these results in (5), we have

$$M'' \cos(x+\psi) - 2M' \sin(x+\psi)(1+\psi') - M \sin(x+\psi)\psi''$$
$$- M\cos(x+\psi)(1+\psi')^2 + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M\cos(x+\psi) = 0. \quad (48)$$

In this the coefficients of $\sin(x+\psi)$ and $\cos(x+\psi)$ must vanish separately; this gives the two equations for determining M and ψ:

$$M'' - M(1+\psi')^2 + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M = 0, \quad (49)$$

$$2M'(1+\psi') + M\psi'' = 0. \quad (50)$$

Equation (50) is readily integrated, giving

$$1+\psi' = \frac{c_1}{M^2}. \quad (51)$$

When we substitute this value of $1+\psi'$ in (49), we have a differential equation for M, namely

$$M'' - \frac{c_1}{M^3} + \left(1+\frac{p}{x^2}+\frac{q}{x^4}\right) M = 0. \quad (52)$$

Now, since we are concerned only with a particular value of M, we may assign to the constant of integration c_1 any value we choose. As we shall see later, it will be most advantageous to take $c_1 = 1$. We know from the equations (19) and from the defining equation for M, namely $M = \sqrt{P^2+Q^2}$, that M must be an even function of x, and moreover that the first term is 1. We therefore assume

$$M = 1 + \frac{k_2}{x^2} + \frac{k_4}{x^4} + \frac{k_6}{x^6} + \dots. \quad (53)$$

Then follow

$$M' = -\frac{2k_2}{x^3} - \frac{4k_4}{x^5} - \frac{6k_6}{x^7} + \dots, \quad (54)$$

$$M'' = \frac{3 \cdot 2k_2}{x^4} + \frac{5 \cdot 4k_4}{x^6} + \frac{7 \cdot 6k_6}{x^8} + \dots. \quad (55)$$

Also, by the binomial theorem,

$$\frac{1}{M^3}=1+\frac{a_2}{x^2}+\frac{a_4}{x^4}+\frac{a_6}{x^6}+\ldots, \tag{56}$$

where the a_2, a_4, a_6, ... are given by the relations:

$$\left.\begin{aligned}
a_2 &= -3k_2, \\
a_4 &= -\frac{4k_2a_2}{2}-3k_4, \\
a_6 &= \frac{-5k_2a_4-7k_4a_2}{3}-3k_6, \\
a_8 &= \frac{-6k_2a_6-8k_4a_4-10k_6a_2}{4}-3k_8, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{57}$$

If now we put the value of M'' from (55), M from (51), and $\frac{1}{M^3}$ from (56) in (52), we have, after putting $c_1=1$,

$$\left.\begin{aligned}
&\qquad\qquad \frac{3.2k_2}{x^4}+\frac{5.4k_4}{x^6}+\frac{7.6k_6}{x^8}+\ldots \\
&+\frac{k_2}{x^2}+\frac{k_4}{x^4}+\frac{k_6}{x^6}+\frac{k_8}{x^8}+\ldots \\
&+\frac{p}{x^2}+\frac{pk_2}{x^4}+\frac{pk_4}{x^6}+\frac{pk_6}{x^8}+\ldots \\
&\qquad\quad +\frac{q}{x^4}+\frac{qk_2}{x^6}+\frac{qk_4}{x^8}+\ldots \\
&-\frac{a_2}{x^2}-\frac{a_4}{x^4}-\frac{a_6}{x^6}-\frac{a_8}{x^8}-\ldots=0.
\end{aligned}\right\} \tag{58}$$

When the coefficients of the various powers of x are put separately equal to zero, the following recurrence formulas are obtained:

$$\left.\begin{aligned}
4k_2 &= -p, \\
4k_4 &= -(p+6)k_2-q+b_4, \\
4k_6 &= -(p+20)k_4-qk_2+b_6, \\
4k_8 &= -(p+42)k_6-qk_4+b_8, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
4k_n &= -[p+(n-1)(n-2)]k_{n-2}-qk_{n-4}+b_n,
\end{aligned}\right\} \tag{59}$$

where $b_4, b_6, b_8, \ldots$ are the same as $a_4, a_6, a_8, \ldots$ as given in equation (57), only with the last term dropped; that is,

$$\left.\begin{aligned}
b_2 &= 0,\\
b_4 &= -\frac{4k_2a_2}{2},\\
b_6 &= \frac{-5k_2a_4 - 7k_4a_2}{3},\\
b_8 &= \frac{-6k_2a_6 - 8k_4a_4 - 10k_6a_2}{4},\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{60}$$

We have then, in (59), a formula for any coefficient in the expansion of M expressed in terms of the preceding coefficients, so that any coefficient can be at once expressed in terms of p and q; naturally, for the later coefficients the calculation becomes somewhat laborious, owing to the complicated character of $a_2, a_4, a_6, \ldots$ when expressed in terms of p and q. Below will be found a list of the first few coefficients calculated in this way.

$$\left.\begin{aligned}
k_2 &= -\frac{p}{4},\\
k_4 &= \frac{5p^2}{32} + \frac{3p}{8} - \frac{q}{4},\\
k_6 &= -\frac{15p^3}{28} - \frac{37p^2}{32} - \frac{15p}{8} + q\left(\frac{5p}{16} + \frac{5}{4}\right),\\
k_8 &= \frac{195p^4}{2048} + \frac{611p^3}{256} + \frac{1821p^2}{128} + \frac{315p}{16} - \frac{45p^2q}{128} - \frac{157pq}{32} + \frac{5q^2}{32} - \frac{105q}{8}.
\end{aligned}\right\} \tag{61}$$

We may use these coefficients of the expansion of M to good advantage in the further computation, particularly in the task of solving by the method of approximation the equation

$$x = \phi - \psi. \tag{39}$$

To show how this may be done, we have from (51), after putting $c = 1$,

$$\psi = \frac{1}{M^2} - 1. \tag{62}$$

Expanding M^{-2} by the binomial theorem, we have

$$\psi = \frac{c_2}{x^2} + \frac{c_4}{x^4} + \frac{c_6}{x^6} + \frac{c_8}{x^8} + \cdots, \tag{63}$$

where c_2, c_4, c_6, ... are given by

$$\left.\begin{aligned} c_2 &= -2k_2, \\ 2c_4 &= -3k_2c_2 - 4k_4, \\ 3c_6 &= -4k_2c_4 - 5k_4c_2 - 6k_6, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{64}$$

If we now integrate (63), putting the constant of integration equal to zero, we have

$$\psi = -\frac{c_2}{x} - \frac{c_4}{3x^3} - \frac{c_6}{5x^5} - \ldots, \tag{65}$$

from which ψ can be expressed in terms of p and q if desired. In order to solve the equation (39), however, we may use the value of ψ as given in (65). We have then to solve

$$x = \phi + \frac{c_2}{x^2} + \frac{c_4}{3x^3} + \frac{c_6}{5x^5} + \frac{c_8}{7x^7} + \ldots. \tag{66}$$

The successive approximations are readily found to be:

$$\left.\begin{aligned} x_1 &= \phi, \\ x_2 &= \phi + \frac{c_2}{\phi}, \\ x_3 &= \phi + \frac{c_2}{\phi} + \frac{\frac{c_4}{3} - c_2^2}{\phi^3}, \\ x_4 &= \phi + \frac{c_2}{\phi} + \frac{\frac{c_4}{3} - c_2^2}{\phi^3} + \frac{2c_2^3 - \frac{4c_2c_4}{3} + \frac{c_6}{5}}{\phi^5}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}\right\} \tag{67}$$

and by means of (64) and (61) these can be expressed in terms of p and q, giving the results previously obtained in (44).

The fact that in (44) the value of the roots of $U_{1a} = 0$ depends only on the undetermined constant α suggests the following method of determining approximately the value of this constant. We have

$$U_{1p} = x\left(A \sin\frac{\sqrt{q}}{x} - B \cos\frac{\sqrt{q}}{x}\right), \tag{28}$$

where A and B have the values given in (30). If we put

$$\left.\begin{aligned} A &= N\cos\theta, \\ B &= N\sin\theta, \end{aligned}\right\} \tag{68}$$

(28) becomes

$$U_{1p} = xN\left(\sin\frac{\sqrt{q}}{x}\cos\theta - \cos\frac{\sqrt{q}}{x}\sin\theta\right), \tag{69}$$

$$U_{1p} = xN\sin\left(\frac{\sqrt{q}}{x} - \theta\right). \tag{70}$$

This will vanish if

$$\sin\left(\frac{\sqrt{q}}{x} - \theta\right) = 0\,; \tag{71}$$

that is, if

$$x = \frac{\sqrt{q}}{\theta + n\pi} \qquad (n = 0, 1, 2, 3, \ldots\ldots). \tag{72}$$

We find now, as in (40),

$$\begin{aligned} \theta = \frac{xf_1}{q^{\frac{1}{2}}} &+ \frac{x^3}{q^{\frac{3}{2}}}\left(f_1f_2 - f_3 - \frac{f_1^3}{3}\right) \\ &+ \frac{x^5}{q^{\frac{5}{2}}}\left(f_5 - f_1f_4 - f_1f_2^2 + f_2f_3 - f_1^3f_2 + f_1^2f_3 + \frac{f_1^5}{5}\right) + \ldots\ldots \end{aligned} \tag{73}$$

We have now to solve approximately the equation

$$x = \frac{\sqrt{q}}{n\pi + \dfrac{xf_1}{q^{\frac{1}{2}}} + \dfrac{x^3}{q^{\frac{3}{2}}}\left(f_1f_2 - f_3 - \dfrac{f_1^3}{3}\right) + \ldots\ldots}, \tag{74}$$

on the supposition that x is small.

It will be more convenient to write (74) in the form

$$x = \sqrt{q}\left(\frac{1}{n\pi} - \frac{xf_1}{q^{\frac{1}{2}}(n\pi)^2} + \frac{f_1^2x^2}{q(n\pi)^3} - \frac{x^3}{q^{\frac{3}{2}}}\left(\frac{f_1^3}{(n\pi)^4} + \frac{f_1f_2 - f_3 - f_1^3}{(n\pi)^2}\right) + \ldots\ldots\right). \tag{75}$$

The successive approximations are:

$$\left.\begin{aligned} x_1 &= \frac{\sqrt{q}}{n\pi} = l_1 \text{ (say)}, \\ x_2 &= \sqrt{q}\left(\frac{1}{n\pi} - \frac{f_1}{(n\pi)^3}\right) = l_2, \\ x_3 &= \sqrt{q}\left(\frac{1}{n\pi} - \frac{f_1}{(n\pi)^3} + \frac{2f_1^2}{(n\pi)^5}\right) = l_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{76}$$

If we put this approximate value of x equal to the approximate value obtained in (44), we have an equation containing only α. This is

$$\alpha + \frac{2n-1}{2}\pi = l_n + \frac{p}{2\left(\alpha + \frac{2n-1}{2}\pi\right)} - \frac{5p^2 + 6p - 12q}{24\left(\alpha + \frac{2n-1}{2}\pi\right)^3} + \dots \quad (77)$$

This gives, as successive approximations for α:

$$\left.\begin{array}{l} \alpha_1 + \frac{2n-1}{2}\pi = l_n, \\ \alpha_2 + \frac{2n-1}{2}\pi = l_n + \frac{p}{2l_n}, \\ \alpha_3 + \frac{2n-1}{2}\pi = l_n + \frac{p}{2l_n} - \frac{11p^2 + 6p - 12q}{24l_n^3}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \quad (78)$$

This approximate value of α, which should hold for certain values of n neither too large nor too small, would indicate that the true value of α probably is a rather complicated function of the two parameters of the equation, p and q.

In a similar way the value of α might be determined so that (44) would give approximately the position of the roots of the other fundamental integral U_{1q}.

IV. *The Roots of* $U'_\alpha = 0$.

In a similar way we may compute the roots of $U'_{1\alpha} = 0$; that is, we may find the maxima and minima of $U_{1\alpha}$. We have

$$U_\alpha = \frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right], \quad (20)$$

where

$$\left.\begin{array}{l} P = 1 - \frac{2^2 f_2}{k^2 e^{2u}} + \frac{2^4 f_4}{k^4 e^{4u}} - \frac{2^6 f_6}{k^6 e^{6u}} + \dots, \\ Q = \frac{2f_1}{ke^u} - \frac{2^3 f_3}{k^3 e^{3u}} + \frac{2^5 f_5}{k^5 e^{5u}} - \dots \end{array}\right\} \quad (79)$$

Now, since U'_α exists, and indeed also in the form of an asymptotic expansion, we may differentiate (20) term by term, the results having of course the same meaning as the original expansion. We thus obtain

$$U'_\alpha = \frac{C}{e^{\frac{u}{2}}}\left[\frac{k}{2}e^u . P\sin\left(\alpha - \frac{ke^u}{2}\right) + \cos\left(\alpha - \frac{ke^u}{2}\right)P' - \frac{Qke^u}{2}\cos\left(\alpha - \frac{ke^u}{2}\right)\right.$$
$$\left. + \sin\left(\alpha - \frac{ke^u}{2}\right) . Q'\right] - \frac{1}{2e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right]. \quad (80)$$

If we put this equal to zero, we have

$$\sin\left(\alpha-\frac{ke^u}{2}\right)\left[\frac{ke^uP}{2}+Q'-\frac{Q}{2}\right]+\cos\left(\alpha-\frac{ke^u}{2}\right)\left[P'-\frac{ke^uQ}{2}-\frac{P}{2}\right]=0. \tag{81}$$

From (79) we have

$$\left.\begin{aligned} P' &= \frac{2^2\,.\,2f_2}{k^2e^{2u}}-\frac{2^4\,.\,4f_4}{k^4e^{4u}}+\frac{2^6\,.\,6f_6}{k^6e^{6u}}+\dots,\\ Q' &= -\frac{2f_1}{ke^u}+\frac{2^3\,.\,3f_3}{k^3e^{3u}}-\frac{2^5\,.\,5f_5}{k^5e^{5u}}+\dots \end{aligned}\right\} \tag{82}$$

For convenience we may return to the former variable, putting, as before, $x=\frac{ke^u}{2}$, so that we have

$$\left.\begin{aligned} P &= 1-\frac{f_2}{x^2}+\frac{f_4}{x^4}-\frac{f_6}{x^6}+\dots,\\ Q &= \frac{f_1}{x}-\frac{f_3}{x^3}+\frac{f_5}{x^5}-\frac{f_7}{x^7}+\dots; \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} P' &= \frac{2f_2}{x^2}-\frac{4f_4}{x^4}+\frac{6f_6}{x^6}-\frac{8f_8}{x^8}+\dots,\\ Q' &= -\frac{f_1}{x}+\frac{3f_3}{x^3}-\frac{5f_5}{x^5}+\frac{7f_7}{x^7}-\dots \end{aligned}\right\} \tag{83}$$

Then the expressions in the square brackets in (81) become

$$\left.\begin{aligned} xP+Q'-\frac{Q}{2} &= x-\frac{f_2+\frac{3}{2}f_1}{x^3}-\frac{f_6+\frac{11}{2}f_5}{x^5}+\dots=R\text{ (say)},\\ P'-xQ-\frac{P}{2} &= -f_1-\frac{1}{2}+\frac{f_3+\frac{5}{2}f_2}{x^2}-\frac{f_5+\frac{9}{2}f_4}{x^4}+\frac{f_7+\frac{13}{2}f_6}{x^6}-\dots=S. \end{aligned}\right\} \tag{84}$$

We put now, for shortness,

$$\left.\begin{aligned} f_1+\frac{1}{2} &= -g_1(p,q),\text{ (or simply }-g_1),\\ f_2+\frac{3}{2}f_1 &= g_2,\\ f_3+\frac{5}{2}\ \dots &\ \dots g_3,\\ f\ \dots & \\ \dots & \end{aligned}\right\} \tag{85}$$

Then the series in (84) become

$$\left.\begin{aligned} R &= x\left[1 - \frac{g_2}{x^2} + \frac{g_4}{x^4} - \frac{g_6}{x^6} + \ldots\right], \\ S &= x\left[\frac{g_1}{x} - \frac{g_3}{x^3} + \frac{g_5}{x^5} - \frac{g_7}{x^7} + \ldots\right]. \end{aligned}\right\} \tag{86}$$

The equation (81) becomes then, after dividing by x,

$$R_1 \sin(\alpha - x) + S_1 \cos(\alpha - x), \tag{87}$$

where

$$\left.\begin{aligned} R_1 &= 1 - \frac{g_2}{x^2} + \frac{g_4}{x^4} - \frac{g_6}{x^6} + \ldots, \\ S_1 &= \frac{g_1}{x} - \frac{g_3}{x^3} + \frac{g_5}{x^5} - \frac{g_7}{x^7} + \ldots \end{aligned}\right\} \tag{88}$$

As in III, we put

$$\left.\begin{aligned} R_1 &= M_1 \cos\psi_1, \\ S_1 &= M_1 \sin\psi_1, \end{aligned}\right\} \tag{89}$$

where M_1 and ψ_1 have exactly the same form as the M and ψ in (37) and (40), except that the g-functions now replace the f-functions. Then (87) becomes

$$M_1 [\sin(\alpha - x)\cos\psi_1 + \cos(\alpha - x)\sin\psi_1] = 0, \tag{90}$$

or

$$\sin(\alpha - x + \psi_1) = 0, \tag{91}$$

which vanishes when

$$\alpha - x + \psi_1 = -n\pi \; (n \text{ an integer or } 0); \tag{92}$$

that is, when

$$x = \alpha + n\pi + \psi_1. \tag{93}$$

We may solve this equation in exactly the same way as in III we solved the equation (39); we need only replace each f-function by the corresponding g-function, due regard being paid to the algebraic signs. We thus obtain

$$x = \alpha + n\pi + \frac{g_1}{[illegible]} \; [illegible] \; g_1^2 \; [illegible] + \ldots \tag{94}$$

If we express this in terms of the f-functions, we have

$$x = \alpha + n\pi - \frac{f_1 + \frac{1}{2}}{\alpha + n\pi} + \frac{f_3 + 2f_2 - f_1 f_2 - 2f_1^2 - \frac{3}{2}f_1 + \frac{f_1^3}{3} - \frac{7}{24}}{(\alpha + n\pi)^3} + \dots, \tag{95}$$

or, in terms of p and q,

$$x = \alpha + n\pi - \frac{\frac{p+1}{2}}{\alpha + n\pi} - \frac{\frac{5p^2}{24} + \frac{7p}{24} + q}{(\alpha + n\pi)^3}$$

$$+ \frac{-\frac{5}{48}p^4 + \frac{103}{960}p^3 - \frac{169}{240}p^2 - \frac{107}{24}p - \frac{113}{480} + q\left(\frac{23}{20}p + \frac{11}{2}\right)}{(\alpha + n\pi)^5} + \dots \tag{96}$$

If we put for x its value $x = \frac{ke^u}{2}$, we have, finally,

$$e^u = \frac{1}{q^{\frac{1}{4}}}\left[\alpha + n\pi - \frac{\frac{p+1}{2}}{\alpha + n\pi} - \frac{\frac{5}{24}p^2 + \frac{7}{24}p + q}{(\alpha + n\pi)^3} + \dots\right]. \tag{97}$$

This formula gives, then, the position of the $(n+1)$th maxima of the curve U_α. Here also a change in the value of the undetermined constant α has practically the effect of displacing the maxima or minima along the axis. A comparison of the equations (96) and (44) shows further that the maxima and minima as given by (96) alternate with the crossings of the axis as given by (44).

The magnitude of the maxima and minima ordinates, or indeed of any ordinate, is of course given by

$$U_\alpha = \frac{C}{e^{\frac{u}{2}}}\left[P\cos\left(\alpha - \frac{ke^u}{2}\right) + Q\sin\left(\alpha - \frac{ke^u}{2}\right)\right], \tag{20}$$

which may be written in the form

$$U_\alpha = \frac{C}{e^{\frac{u}{2}}} M \cos\left(\alpha - \frac{ke^u}{2} - \psi\right), \tag{98}$$

where M is given by (61) as

$$M = 1 - \frac{p}{q^{\frac{1}{2}}e^{2u}} + \frac{\frac{5}{2}p^2 + 6p - 4q}{qe^{4u}} + \dots \tag{99}$$

and ψ is given by

$$\psi = \tan^{-1}\left(\frac{p}{q^{\frac{1}{4}}e^u} + \frac{\frac{p^3}{3} - \frac{p^2}{3} - 2p + 8q}{q^{\frac{3}{4}}e^{3u}}\right.$$

$$\left. + \frac{\frac{13}{60}p^5 - \frac{5}{12}p^4 - \frac{p^3}{10} - \frac{163}{15}p^2 + \frac{316}{3}p + q\left(8p^2 - \frac{104}{5}p - 96q\right)}{q^{\frac{5}{4}}e^{5u}} + \dots\right) \tag{100}$$

The length of any ordinate is then given by (98), thus depending upon both the undetermined constants C and α. This formula also shows how, for increasing u, the lengths of the maximum and minimum ordinates continually decrease.

V. *Numerical Verification.*

The equation considered by Heine, Dannacher and Butts is written in the form

$$\frac{d^2E}{d\phi^2}+\left(\frac{8}{b}\cos 2\phi+4z\right)E=0. \tag{101}$$

If we put $\phi=iu$, we have

$$\frac{d^2E}{du^2}-\left(\frac{8}{b}\cosh 2u+4z\right)E=0. \tag{102}$$

If now we make the following substitutions:

$$\left.\begin{aligned}&\cosh 2u=2\cosh^2u-1,\\&\frac{16}{b}=k^2,\\&\mathfrak{B}+\frac{8}{b}=4z,\end{aligned}\right\} \tag{103}$$

equation (102) becomes

$$\frac{d^2E}{du^2}-(k^2\cosh^2u+\mathfrak{B})E=0, \tag{104}$$

which differs from equation (1) only in the sign of the second term. Now, by means of transformations similar to those employed in I, we reduce the equation (104) to the following form:

$$\frac{d^2E_1}{dx^2}-\left(1-\frac{r}{x^2}-\frac{s}{x^4}\right)E_1=0, \tag{105}$$

where

$$\left.\begin{aligned}r&=-\frac{1}{4}-\frac{k^2}{2}-\mathfrak{B},\\s&=-\frac{k^4}{16}.\end{aligned}\right\} \tag{106}$$

For large values of x, (105) becomes approximately

$$\frac{d^2E_1}{dx^2}-E_1=0, \tag{107}$$

so that we may assume as an asymptotic solution of (105), for large values of x,

$$E_1 = e^x\left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \dots\right). \tag{108}$$

Proceeding now as in I, we find, for the semi-convergent solution of (102) for large values of x,

$$E = A_0 \frac{e^{\frac{ke^u}{2}}}{e^{\frac{u}{2}}}\left[1 + \frac{h_1}{\frac{ke^u}{2}} + \frac{h_2}{\left(\frac{ke^u}{2}\right)^2} + \frac{h_3}{\left(\frac{ke^u}{2}\right)^3} + \dots\right], \tag{109}$$

where A_0 is an undetermined constant, and where the h_1, h_2, h_3, ... are abbreviations for the following:

$$\left.\begin{aligned} h_1 &= \frac{r}{2}, \\ h_2 &= \frac{r(r+2)}{2^2 \,.\, 2\,!}, \\ h_3 &= \frac{r(r+2)(r+6)}{2^3 \,.\, 3\,!} + \frac{s}{2 \,.\, 3}, \\ &\dots\dots\dots\dots\dots, \\ h_n &= \frac{h_{n-1}[r + n(n-1)] + h_{n-3} \,.\, s}{2(n-1)}; \end{aligned}\right\} \tag{110*}$$

and where

$$\left.\begin{aligned} r &= -\frac{1}{4} - \frac{k^2}{2} - \mathfrak{B}, \\ s &= -\frac{k^4}{16}. \end{aligned}\right\} \tag{111*}$$

Also between the constants of equation (102) and (109) there exist the following relations:

$$\left.\begin{aligned} r &= -\frac{1}{4} - 4z, \\ s &= -\frac{16}{b^2}, \\ k &= \frac{4}{\sqrt{b}}. \end{aligned}\right\} \tag{112}$$

* It may be remarked that $h_1, h_2, h_3, \dots$ are the same as $f_1, f_2, f_3, \dots$, as defined by (19), except that here all the signs are positive. Also $p = -r$ and $q = -s$.

A solution of (102) as given by Heine is

$$E(u) = \frac{a_0}{2} + a_1 \cosh 2u + a_2 \cosh 4u + \ldots, \tag{113}$$

where $a_0 = 1$, $a_1 = \frac{1}{2} bz$, and the other a's are given by the recurrence formula

$$a_{n+1} = b(n^2 - z)a_n - a_{n-1}. \tag{114}$$

This series (113) converges (for any assumed value of b) only for values of z which are the roots of a certain transcendental function.* For large values of u it at first diverges rapidly, and ultimately converges slowly. The series given by (109), on the other hand, at first converges rapidly, for reasonably large values of u, and then diverges rapidly. It may be used for approximate numerical calculation, provided care is taken not to include any terms after divergence begins. We wish to show by means of a numerical example how the asymptotic expansion given in (109) leads to the same result as (113), but with much less labor in computation.

It has been shown by Butts† that if b be taken equal to .1, then one root of the *limiting function*, that is, one value of z for which the series (113) converges, is 5.58134 (correct to five decimal places). He gives the values of the coefficients computed for this value of b and z as

$$\left.\begin{aligned} a_0 &= 1.0000, \\ a_1 &= -0.2790, \\ a_2 &= -0.8721, \\ a_3 &= 0.4169, \\ a_4 &= 1.0147, \\ a_5 &= 0.6402, \\ a_6 &= 0.2284, \\ a_7 &= 0.0548, \\ a_8 &= 0.0096, \\ a_9 &= 0.0015, \\ a_{10} &= 0.0001. \end{aligned}\right\} \tag{115}‡$$

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* See Dannacher, *l. c.*, p. 29.

† W. H. Butts., *l. c.*, p 20.

‡ a_9 and a_{10} are not exactly as given by Butts, but have been changed slightly by means of Heine's formula $a_{n+1} < \frac{a_n}{bn^2}$, which holds from a certain n on.

If we put now $u = .7$, equation (113) gives us

$$\begin{aligned} E(.7) = .5 - .2790 \cosh 1.4 - .8721 \cosh 2.8 + .4169 \cosh 4.2 \\ + 1.0147 \cosh 5.6 + .6402 \cosh 7.0 + .2284 \cosh 8.4 \\ + .0547 \cosh 9.8 + .0096 \cosh 11.2 + .0015 \cosh 12.6 \\ + .0001 \cosh 14. \end{aligned} \tag{116}$$

With a four-place logarithm table this gives

$$\begin{aligned} E(.7) = .5000 - .6000 - 7.7980 + 13.9100 + 136.9000 + 351.0000 \\ + 507.3000 + 494.2000 + 354.2000 + 222.5000 + 60.7500 = 2133. \end{aligned} \tag{117}$$

In order to use equation (109) we must in (112) substitute the values of b and z, namely .1 and 5.58134. This gives

$$\left.\begin{aligned} r &\doteq -22.47536, \\ s &= -1600, \\ k &= 4\sqrt{10}. \end{aligned}\right\} \tag{118}$$

Substituting these in (109), putting $u = .7$, and using four-place logarithms as before, we have, using four terms of the series,

$$\begin{aligned} E(.7) = 239500 A_0[1 - .8824 + .3546 - (.1291 + .0765) \\ + (.0079 + .0138 + .0853)] = 239500 A_0(.3727). \end{aligned} \tag{119}$$

A comparison of (117) and (119) serves now to determine A_0. We have, using four-place logarithms as before,

$$A_0 = .02388. \tag{120}$$

This value of A_0 may now be employed in finding the value of $E(u)$ for other arguments, by means of the series (109). For example, if we put $u = .8$ and use, as before, four terms of the series and four-figure logarithms, we have

$$\begin{aligned} E(.8) = 869600 A_0[1 - .7976 + .2904 - (.0956 + .0566) \\ + (.0052 + .0089 + 0572)] = 869600(.02388)(.4119) = 8552. \end{aligned} \tag{121}$$

We may verify this by means of (113). Carrying the series to ten terms and using, as before, four-place logarithms, we have

$$\begin{aligned} E(.8) = .5000 - .8672 - 8.5130 + 25.3400 + 305.20 + 954.2 \\ + 1766 + 2004 + 1599 + 1345 + 444.0 = 8434. \end{aligned} \tag{122}$$

Considering the roughness of the approximation used, and particularly the possible incorrectness of a_6, a_7, a_8, a_9, a_{10}, this may be regarded as a very close

agreement. The fact that (122) is too small may be accounted for by the fact that the next few terms would contribute materially to the result. In using series (113) for large values of the argument it would be necessary to know the value of z and the corresponding values of $a_1, a_2, \ldots$ with much greater exactness. This fact determined in a way the choice of the particular values $u = .7$ and $u = .8$ in this illustration. For values of u much smaller than .7, (109) does not converge at all; and for values of u much greater than .8, the use of (113) is too uncertain. That the example is a particularly unfavorable one is due to the fact that b was chosen so small and z large in comparison, which has the effect of making both r and s large. Under more favorable circumstances (109) would converge much more rapidly, and for smaller values of the argument.*

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*Since the above was written, my attention has been called to an article by R. C. Maclaurin in Vol. XVII, Pt. I, of the *Transactions of the Cambridge Philosophical Society*. In this article, which is entitled "On the Solutions of the Equation $(\Delta^2 + k^2)\psi = 0$ in Elliptic Coordinates and their Physical Applications," Mr. Maclaurin reduces the equation $(\Delta^2 + k^2)\psi = 0$ to the form

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - p^2)y = 0.$$

He then obtains power-series solutions in the neighborhood of $-1, +1, \infty$, this latter being asymptotic. He suggests the possibility of using this asymptotic expansion in finding expressions for the roots of the function itself and of its first derivative, and calls attention to the importance of these roots in practical applications.

A Theory of Invariants.

BY LEONARD EUGENE DICKSON.

General Theory, §§ 1–6.

1. Consider a system S of forms $f_1, \ldots, f_s$ on m variables, where f_i is the general polynomial of degree d_i, having as coefficients arbitrary parameters in any given field F, finite or infinite. Let L be any given group of m-ary linear homogeneous transformations with coefficients in F. The particular systems S', S'', .., obtained from the general system S by assigning to the coefficients particular sets of values in the field, may be separated into classes C_i under the group L, such that S' and S'' belong to the same class if and only if they are equivalent under L.

We shall employ the term function in Dirichlet's sense of correspondence and shall consider only single-valued functions taking exclusively values in the field F.

Let the function ϕ have one and only one value in field for each of the systems S', S'', ... In particular, if, for each i, ϕ has the same value v_i for all the systems in the class C_i, then ϕ is an invariant under the group L.

For a finite field such an invariant is a rational integral function of the coefficients of S, an explicit expression for which is given in § 4. More convenient expressions may be given when S is a special system (§§ 10–18).

2. Let the invariants $I_1, \ldots, I_r$ completely characterize the classes C_i, *i. e.*, each I_k has the same value for two classes only when the latter are identical. Let ϕ be any invariant of the kind defined in § 1. Then ϕ takes one and only one value for each class, while there is one and only one class for each set of values of the I_k. Hence ϕ is a single-valued function of $I_1, \ldots, I_r$, as related.

In particular, let ϕ be a rational integral function of the coefficients of S. Let ϕ be of degree α_{ij} in the coefficient a_{ij}. Then ϕ is completely determined by $N = \Pi(\alpha_{ij}+1)$ distinct* sets of values of the a_{ij} in the field. By means of

* For the $GF[p^n]$, we may set $\alpha_{ij} \leqq p^n - 1$.

the resulting N sets (not necessarily distinct) of values of $I_1, .., I_r$, we may construct by a general interpolation formula (§ 3) a polynomial in $I_1, .., I_r$ which takes the same values as ϕ for the N sets. *Hence any rational integral invariant is a rational integral function of* $I_1, .., I_r$.

3. *Interpolation for Any Number of Variables.* We construct a polynomial in $x, y, z, ..$, which takes prescribed values for a finite number of distinct sets of values of the variables. Consider first the case of two variables. Let x take the distinct values $x_1, .., x_r$. When $x = x_i$, let y take the distinct values $y_{i1}, .., y_{is_i}$. We desire a polynomial $P(x, y)$, which shall take the value v_{ij} when $x = x_i$, $y = y_{ij}$, for each set of subscripts i, j. By any interpolation formula in one variable y, construct for each i a polynomial $Y_i(y)$ which takes the value v_{ij} for $y = y_{ij}$ $(j = 1, .., s_i)$. For example, Lagrange's formula gives

$$Y_i(y) = \sum_{j=1}^{s_i} v_{ij} \prod_{J=1, .., s_i;\, J \neq j} \left(\frac{y - y_{iJ}}{y_{ij} - y_{iJ}} \right). \tag{1}$$

Next, we construct a polynomial P in x which becomes Y_i for $x = x_i$. If we again use Lagrange's formula, we have

$$P(x, y) = \sum_{i=1}^{r} Y_i(y) \prod_{I=1, .., r;\, I \neq i} \left(\frac{x - x_I}{x_i - x_I} \right). \tag{2}$$

When the function (1) is inserted into (2), we obtain the required interpolation formula for two variables. The extension to three or more variables is obvious. Speaking geometrically, we first interpolate for the points in each line parallel to one of the Cartesian axes of coordinates, then for the various lines in a plane parallel to a coordinate plane, etc.

4. An important case is that in which each variable ranges independently over the p^n elements e_i of the Galois field $GF[p^n]$. Then* in (1) the product may be given the form $(y^{p^n} - y)/(e_j - y)$. Thus

$$P(x, y) = \sum_{i, j=1}^{p^n} v_{ij} \frac{(x^{p^n} - x)(y^{p^n} - y)}{(e_i - x)(e_j - y)}. \tag{3'}$$

In general, the integral function obtained from

$$P(x_1, .., x_s) = \sum_{i_1, .., i_s}^{1, .., p^n} v_{i_1, .., i_s} \frac{(x_1^{p^n} - x_1)..(x_s^{p^n} - x_s)}{(e_{i_1} - x_1)..(e_{i_s} - x_s)}, \tag{3}$$

takes the value $v_{i_1, .., i_s}$ for $x_1 = e_{i_1}, .., x_s = e_{i_s}$. Any polynomial, of degree $\leqq p^n - 1$ in each x, which takes these values v is identical with P.

* Cf. *Annals of Mathematics*, 1897, p. 69.

For a system S of forms in a finite field the investigation of invariants, which take one and but one value in the field for each set of values of the coefficients, may therefore be restricted to polynomials in the coefficients. The *characteristic** invariant I_k for the class C_k is that invariant which has the value unity for the class C_k and the value zero for the remaining classes. Let $a_1, \ldots, a_s$ be the coefficients in the system S; $a_1^{(k)}, \ldots, a_s^{(k)}$ the coefficients in a system of forms in the class C_k. Then by (3),

$$I_k = \sum_{a_1^{(k)}, \ldots, a_s^{(k)}} \prod_{j=1}^{s} \frac{a_j^{p^n} - a_j}{a_j^{(k)} - a_j}. \tag{4}$$

In particular, for the class C_0, composed of the system of forms with all coefficients zero, the characteristic invariant is

$$I_0 = \prod_{j=1}^{s} (1 - a_j^{p^n - 1}). \tag{4'}$$

From the definition of the I_k or from their expressions (4), we have

$$I_k^2 = I_k, \quad I_k I_l = 0 \ (l \neq k), \quad \sum_{k=0}^{f-1} I_k = 1, \tag{5}$$

where f denotes the number (necessarily finite) of the classes.

Any invariant I takes certain values $v_0, \ldots, v_{f-1}$ for the classes $C_0, \ldots, C_{f-1}$, so that $I = \Sigma v_k I_k$. *Any invariant can be expressed in one and but one way as a linear homogeneous function of the characteristic invariants. The number of linearly independent invariants equals the number of classes.*

The fact that any invariant is a linear function of the I_k also follows from (5) and the general theorem in § 2, since the present set of I_k is a special case of the set there discussed.

5. When the group L is the group G of all m-ary linear homogeneous transformations in the field F, the invariants, defined in § 1, are *absolute* invariants of the system S. When L is the group G_1 of all the transformations of determinant unity, those invariants of G_1 which are multiplied by Δ^w under any transformation of determinant Δ are *relative* invariants of *weight* w of the system S.

The theory of the characteristic invariants enables us to prove (*Transactions, l. c.*) that *when F is the $GF[p^n]$, the number of linearly independent invariants, relative and absolute, equals the number of classes under the group G_1.*

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* *Transactions Amer. Math. Soc.*, Vol. X (1909), p. 126. The existence of the invariants was there established, but not the explicit formulæ (3), (4).

In the special systems investigated below, we do not presuppose this result, but obtain a direct verification of it.

6. In the investigation of the invariants of a system of forms in a field F under a group L, the chief aim is the determination of a fundamental system of independent invariants* (all invariants being single-valued functions of these) and a complete set of independent relations between them. A practical method of determining these relations for the case of a finite field may be based upon the knowledge of a complete set of linearly independent invariants. As we shall usually not go to the trouble (in general very great) to determine† the characteristic invariants, we propose the following procedure‡ for the determination of a complete set of linearly independent invariants, which establishes automatically their linear independence.

Let the classes be separated into sets $K_1, K_2, \ldots, K_l$ (in practice by specifying the ranges of certain parameters) and let there be associated with K_i a set V_i of linearly independent§ invariants, in number equal to the number of classes in K_i, and such that every invariant in the set V_i vanishes for all sets K_j $(j > i)$. The aggregate of the invariants in the sets V_i constitutes a complete system of linearly independent invariants. Indeed, their number is the total number of classes, while any linear relation has its coefficients all zero, as shown by employing in turn the classes in $K_l, K_{l-1}, \ldots, K_1$. For instance, for the classes in K_l, the invariants in $V_1, \ldots, V_{l-1}$ vanish, and those in V_l are linearly independent.

In particular, if L is the group G_1 (§ 5), we obtain a complete set of linearly independent relative and absolute invariants, when the chosen invariants of the sets V_i are of that kind.

Field‖ C of all Real and Complex Numbers.

7. Consider a single quadratic form on m variables. Denote its determinant by D. Within the group G_1, a complete set of canonical types is

$$Dx_1^2 + x_2^2 + \ldots + x_m^2 \ (D \neq 0), \quad x_1^2 + \ldots + x_i^2 \ (i = 1, \ldots, m-1), \quad 0.$$

* To be chosen on the basis of their expressing fundamental properties of the system of forms, rather than the simplicity of their expressions or the smallness of their number.

† More explicitly than by the general expression (4).

‡ First employed in my paper on the modular invariants of the general system of q linear forms in m variables, *Proc. Lond. Math. Soc.*, 1909.

§ Tested by employing the classes in K_i alone.

‖ The discussion in §§ 7–9 applies also to the infinite field F_p given by the aggregate of the Galois fields of orders p^n, $n = 1, 2, 3, \ldots$, provided $p < m$ (also to certain fields contained in F_p).

The classes are completely characterized by the invariants D and r, where r is the rank of the matrix of the form. Thus D and r form a fundamental system of invariants. A complete set of independent relations is

$$r(r-1)\ldots(r-m)=0, \quad rD=mD.$$

A polynomial in D and r therefore equals the sum of a polynomial in D and a polynomial in r. Thus (§ 2) the only rational integral invariants are the polynomials in D. The characteristic invariant for $x_1^2+\ldots+x_i^2\,(i<m)$, including 0 for $i=0$, is

$$\Pi\left(\frac{r-t}{i-t}\right) \quad (t=0,1,\ldots,m;\ t\neq i).$$

8. A binary cubic form in C can be transformed within G_1 into one of the canonical types x^3, $xy(x+ry)$, 0, where r is a particular square root of the discriminant D. The corresponding classes may be designated T, C_D, V. Let m be the function of the coefficients which specifies the maximum multiplicity of a linear factor, viz., $m=3,2,1,0$ for classes T, C_0, $C_D\,(D\neq 0)$, V, respectively. Hence D and m form a fundamental system of invariants. A complete set of independent relations is

$$m(m-1)(m-2)(m-3)=0, \quad mD=D.$$

Thus $P(D,m)=P_1(D)+P_2(m)$, where the P's denote polynomials. Hence (§ 2) the only rational invariants are $P_1(D)$. The characteristic invariants for T, C_0, V are

$$\tfrac{1}{6}m(m-1)(m-2), \quad -\tfrac{1}{2}m(m-1)(m-3), \quad -\tfrac{1}{6}(m-1)(m-2)(m-3).$$

9. For the field C we consider a pair of binary quadratic forms with the determinants a, b, bilinear invariant θ, and resultant R (§ 13).

If $R\neq 0$, the family has two independent unary forms, which become multiples of x^2 and y^2 under a transformation of G_1. Then

$$q_1=a_0x^2+a_2y^2, \quad q_2=b_0x^2+b_2y^2, \quad R=-(a_0b_2-a_2b_0)^2\neq 0.$$

Applying $(y,-x)$ if necessary, we may set $a_0\neq 0$. Applying $(\rho x, \rho^{-1}y)$, where $\rho^2=a_0^{-1}$, we obtain a similar pair with $a_0=1$, viz.,

$$Q_1=x^2+ay^2, \quad Q_2=B_0x^2+B_2y^2, \quad B_0B_2=b, \quad B_2+aB_0=\theta.$$

If $a=0$, then $\theta^2=-R$, $\theta\neq 0$, $B_2=\theta$, $B_0=b/\theta$. If $a\neq 0$,

$$B_0=(\theta\pm\sqrt{-R})/(2a), \quad B_2=(\theta\mp\sqrt{-R})/2.$$

But the two resulting pairs of forms are interchanged by $(a^{1/2}y, -a^{-1/2}x)$. Hence the invariants a, b, θ completely characterize the classes with $R \neq 0$ under G_1.

If the rank r_1 of q_1 is zero, so that $q_1 \equiv 0$, the classes are characterized by $r_1 = 0$ and the values of b, r_2 (§ 7). For use in characterizing* the classes in which $q_1 \not\equiv 0$, $q_2 = cq_1$, let $M = c$ in that case, but $M = 0$ if $q_1 \equiv 0$ or if $q_1 \not\equiv 0$ and q_2 is not a multiple of q_1. According to the definition in § 1, M is an invariant. We may also define M by means of the pencil $\lambda q_1 + \mu q_2$, λ and μ being arbitrary variables; if the functions $\lambda a_i + \mu b_i$ have a common divisor of the form $\lambda + c\mu$, we take $M = c$; in the contrary case, we take $M = 0$.

It remains to consider the case $R = 0$, $M = 0$, $r_1 > 0$, $r_2 > 0$, so that neither q_1 nor q_2 vanishes identically, and q_2 is not a multiple of q_1. Then

$$d = a_0 b_1 - a_1 b_0, \quad e = a_0 b_2 - a_2 b_0, \quad f = a_1 b_2 - a_2 b_1 \tag{6}$$

are not all zero, viz., the Jacobian of q_1 and q_2 is not identically zero. We have the relations

$$R = 4df - e^2, \quad a_0 f - a_1 e + a_2 d = 0. \tag{7}$$

Multiply (7_1) by a_0^2 and eliminate f by (7_2). Then since $R = 0$,

$$\delta^2 = -4ad^2, \quad \delta \equiv a_0 e - 2a_1 d. \tag{8}$$

For the present case, a and b are not both zero.

(i) $a \neq 0$. First, let $a_0 \neq 0$. Then $d \neq 0$. Let $r = \delta/(2d)$. Then r is a particular square-root of $-a$. Applying the transformation of determinant unity,

$$x = \frac{r + a_1}{a_0} X + \frac{r - a_1}{2r} Y, \quad y = -X + \frac{a_0}{2r} Y, \tag{9}$$

we get (since $r^2 = -a$)

$$q_1 = 2rXY, \quad q_2 = \beta_0 X^2 - r^{-1}\theta XY + \beta_2 Y^2, \tag{10}$$

$$\beta_0 = (\delta - 2rd)/a_0^2, \quad \beta_2 = (\delta + 2rd)/(4r^2). \tag{11}$$

Inserting the special value of r, we have $\beta_0 = 0$, $\beta_2 \neq 0$. Applying a transformation (ρ, ρ^{-1}), we obtain

$$Q_1 = 2rXY, \quad Q_2 = -r^{-1}\theta XY + Y^2. \tag{12}$$

Next, let $a_0 = 0$, so that $a_1 \neq 0$. Eliminating e from (7), we get

$$dk = 0, \quad k \equiv 4a_1^2 f - a_2^2 d.$$

The factors d, k are not both zero. If $d = 0$, so that $b_0 = 0$, the transformation of determinant unity

$$x = X - \tfrac{1}{2} a_1^{-1} a_2 Y, \quad y = Y \tag{13}$$

* It suffices to employ M alone if $M \neq 0$, but $r_1 > 0$, a, $r_2 = 0$, if $M = 0$.

replaces q_1, q_2 by (10) with $r=a_1$, $\beta_0=b_0$, $\beta_2=k/(4a_1^3)\neq 0$. If $d\neq 0$, so that $k=0$, the transformation

$$x=\tfrac{1}{2}a_1^{-1}a_2X+Y,\quad y=-X$$

replaces q_1, q_2 by (10) with $r=-a_1$, $\beta_0=k/(4a_1^3)$, $\beta_2=b_0\neq 0$. In the last two cases we obtain (12) with $r=a_1$ and $-a_1$, respectively.

In every case we have obtained the canonical type (12) in which r is a particular square-root of the invariant $-a$. No transformation of G_1 replaces (12) to a like pair with the parameter $-r$. The additional invariant V_a necessary to characterize the classes may be defined by its values as follows: If $R\neq 0$, or $M\neq 0$, or $r_2=0$, or $a=0$, set $V_a=0$. If $R=M=0$, $r_2>0$, $a\neq 0$, set

$$V_a=\delta/(2d) \text{ for } a_0\neq 0,\quad V_a=a_1 \text{ for } a_0=d=0,\quad V_a=-a_1 \text{ for } a_0=0,\ d\neq 0.$$

Since V_a has the same value for all sets of forms in a class, it is an invariant. For the corresponding rational integral invariant in a finite field, see § 14.

(ii) $b\neq 0$. We employ the invariant V_b derived from V_a by interchanging the a's and b's. By modifying our definition of V_a when $a=0$, we could make a single invariant cover both cases $a\neq 0$ and $a=0$, $b\neq 0$.

We have now shown that a complete system of invariants is given by

$$a,\ b,\ \theta,\ r_1,\ r_2,\ M,\ V_a,\ V_b. \tag{14}$$

Invariants of Two Binary Quadratic Forms in the $GF[2^n]$.

10. Since the modulus is 2, the forms are designated

$$q_1=a_0x^2+a_1xy+a_2y^2,\quad q_2=b_0x^2+b_1xy+b_2y^2. \tag{15}$$

A single form q_1 has the invariants,* the third being (4′),

$$a_1,\quad H_a=\chi(a_0a_1^ea_2),\quad I_a=\prod_{i=0,1,2}(1-a_i^m), \tag{16}$$

where $e=2^n-3$ if $n>1$, $e=1$ if $n=1$, while

$$m=2^n-1,\quad \chi(s)=\sum_{i=0}^{n-1}s^{2^i}.$$

The necessary and sufficient condition that q_1 be irreducible in the field is $H_a=1$; that it be the product of two distinct linear factors, $H_a=0$, $a_1\neq 0$; that it be a perfect square, $a_1=I_a=0$; that it vanish identically, $I_a=1$ Within the group G_1, the corresponding canonical forms are

$$Q_1=a_1^2x^2+a_1xy+cy^2,\quad a_1xy,\quad x^2,\quad 0, \tag{17}$$

* Concerning the invariants here employed, see AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), pp. 109, 115. This paper is cited henceforth as *A. J.*

where $a_1 \neq 0$ and c is a particular solution of $\chi(c) = 1$. Hence, by § 2, the three invariants (16) form a fundamental system of invariants of q_1.

When q_1 is transformed within G_1 into one of the types (17), let q_2 become

$$Q_2 = B_0 x^2 + b_1 xy + B_2 y^2.$$

For (17_1) it suffices to normalize

$$Q_2 - a_1^{-1} b_1 Q_1 = l^2, \quad l \equiv rx + sy.$$

For the eliminant of Q_1, l and the resultant of q_1, q_2, we have

$$E = a_1^2 s^2 + a_1 rs + cr^2, \quad R = E^2.$$

If $E = 0$, then $r = s = 0$ by the irreducibility of Q_1, so that the forms are already in their canonical form (I). If $E \neq 0$,

$$\begin{pmatrix} ksa_1 & krca_1^{-1} \\ kra_1 & k(r + sa_1) \end{pmatrix}, \quad k \equiv E^{-1/2}$$

is an automorph of Q_1, has determinant unity, and replaces l by $E^{1/2} a_1^{-1} y$. Hence, if q_1 is irreducible, the canonical type is

(I) $Q_1 = a_1^2 x^2 + a_1 xy + cy^2, \quad Q_2 = a_1^{-1} b_1 Q_1 + a_1^{-2} R^{1/2} y^2 \quad (a_1 \neq 0).$

As in *A. J.*, § 18, § 19 (with $k = 1$), the remaining canonical types are

(II), (III) $Q_1 = a_1 xy, \quad Q_2 = x^2 + b_1 xy + a_1^{-2} R y^2$, or $b_1 xy \quad (a_1 \neq 0)$;
(IV) $Q_1 = x^2, \quad Q_2 = b_1^2 c R^{-1/2} x^2 + b_1 xy + R^{1/2} y^2 \quad (b_1 R \neq 0)$;
(V), (VI) $Q_1 = x^2, \quad Q_2 = b_1 xy + R^{1/2} y^2$ (b_1, R not both 0), or $b_0 x^2$;
(VII)–(X) $Q_1 = 0, \quad Q_2 = b_1^2 x^2 + b_1 xy + cy^2, \ b_1 xy, \ x^2, \ 0 \quad (b_1 \neq 0).$

To construct the additional invariant required by the type VI, we may proceed as with M in § 9, or as follows. The relation

$$I_{a+kb} = I_a + I_a(I_b + 1) k^m + \sum_{r=1}^{m} k^r Z_r \quad (k^{m+1} \equiv k)$$

defines certain absolute invariants Z_r. But $Z_r = Z_1^r$, as shown in *A. J.*, § 21; while Z_1 has the following characteristic properties: If q_1 is not identically zero, $Z_1 = t$ when $q_2 \equiv tq_1$, $Z_1 = 0$ when q_2 is not a constant multiple of q_1; if $q_1 \equiv 0$, then $Z_1 = 0$. Thus $Z_1 = (1 - R^m) a_1^{-1} b_1$ for type I, $a_1^{-1} b_1$ for III, b_0 for VI, zero for the remaining types.

We thus employ Z_1 to differentiate the types VI from each other, and type II with $R = 0$, $b_1 \neq 0$ from type III with $b_1 \neq 0$. The invariants

$$a_1, \ H_a, \ I_a, \ b_1, \ H_b, \ I_b, \ R, \ Z_1 \tag{18}$$

completely characterize the types I, .., X and hence (§ 2) *form a fundamental system of invariants of forms* (15) *in the* $GF[2^n]$.

These eight invariants are independent (*A. J.*, § 23). By employing invariants whose interpretation is not so immediate, we may obtain (*A. J.*, § 32) a fundamental system of six invariants (four if $n=1$).

11. THEOREM. *The* $2^{3n+1}+2^{2n}$ *invariants** *in the accompanying table form a complete set of linearly independent invariants of two binary quadratic forms in the* $GF[2^n]$.

The following table is arranged according to the principle in § 6. The number N of canonical types specified in any line equals the number of invariants in that line, the linear independence of the latter being readily established by inserting the values of the invariants for the forms in that line. The linear independence of all the invariants then follows from the fact that each invariant vanishes for all the types in the later lines of the table.

PAIRS OF FORMS.	INVARIANTS.	NUMBER.
I, $H_b=1$	$H_a H_b a_1^i b_1^j R^e$	$m^2\, 2^{n-1}$
I, $H_b=0,\ b_1\neq 0$	$H_a a_1^i b_1^{j+1} R^e$	$m^2\, 2^{n-1}$
I, $b_1=0$	$H_a a_1^i R^j,\ H_a a_1^i I_b$	$m\, 2^n$
III	$a_1^{i+1} I_b,\ a_1^{i+1} Z_1,\ a_1^i b_1^j Z_1\ (j>0)$	$m\, 2^n$
II, $H_b=1$	$H_b a_1^{i+1} b_1^j R^e$	$m^2\, 2^{n-1}$
II, $H_b=0,\ b_1\neq 0$	$a_1^{i+1} b_1^{j+1} R^e$	$m^2\, 2^{n-1}$
II, $b_1=0$	$a_1^{i+1} R^j,\ a_1^{i+1} R^m$	$m\, 2^n$
VII	$I_a b_1^i H_b$	m
VIII	$I_a b_1^{i+1}$	m
X	$I_a I_b$	1
IX	I_a	1
IV	$H_b b_1^i R^j$	m^2
V, $R\neq 0$	$b_1^i R^{j+1},\ b_1^m R^{j+1}$	$m\, 2^n$
V, $R=0$	b_1^{i+1}	m
VI	$I_b,\ Z_1^j$	2^n

Throughout the table the exponents have the ranges

$$i, j=0, 1, \ldots, m-1;\ e=0, 1, \ldots, 2^{n-1}-1.$$

* Identical with the set (82) of *A. J.*, § 28, there proved complete for $n\leq 3$. For $n=1$, we may delete $a_1 Z_1$ from the fourth line of the table and insert Z_1, in view of (69), *A. J.*

On account of the complexity of the expressions for H_b, we have made three subdivisions for types I and II. Since $\chi(c)=1$, $\chi^2=\chi$, we have for I,

$$H_b=\chi(cb_1^m)+\chi(a_1^{-1}b_1^{2^n-2}R^{1/2})=b_1^m+\chi(a_1^{-2}b_1^{2^n-3}R),$$

the final term being also the value of H_b for type II. Hence in lines 1, 2, 5, 6 of the table, R satisfies an equation of degree 2^{n-1}.

Relations (68)–(73) of *A. J.*, § 25, together with $b_1Z_1=a_1Z_1$ for $n=1$, enable us to express any product of the invariants (18) as a linear function of those in the table.

One Binary Quadratic Form in the $GF[p^n]$, $p>2$.

12. For a single form in the $GF[p^n]$, $p>2$,

$$q_1=a_0x^2+2a_1xy+a_2y^2 \quad (a=a_0a_2-a_1^2), \tag{19}$$

the canonical types within the group G_1 are

$$x^2+ay^2 \ (a\neq 0),\ x^2,\ \nu_1x^2,\ 0 \quad (\nu_1 \text{ a fixed not-square}), \tag{20}$$

according as respectively $a\neq 0$, $-Q=$ square, $-Q=$ not-square, or $a=Q=0$, where Q denotes the absolute invariant (*A. J.*, § 7):

$$Q=(a_0^\tau+a_2^\tau)\Big\{\sum_{i=0}^{\tau}a_0^ia_2^ia_1^{2\tau-2i}-1\Big\} \quad [\tau=\tfrac{1}{2}(p^n-1)]. \tag{21}$$

Hence a and Q form a fundamental system (§ 2). The identically vanishing type may be characterized by $I=1$, where

$$I=1-a^{2\tau}-Q^2=(1-a_0^{2\tau})(1-a_1^{2\tau})(1-a_2^{2\tau}). \tag{22}$$

A complete set of linearly independent invariants is (§ 6)

$$a^i \ (i=1,\ldots,2\tau),\ Q,\ Q^2,\ 1;$$

any relation between them follows from

$$a^{2\tau+1}=a,\quad aQ=0,\quad Q^3=Q.$$

The last two are proved by multiplying (22) by a and Q, respectively.

Two Binary Quadratic Forms in the $GF[p^n]$, $p>2$, §§ 13–17.

13. Consider a pair of forms defined by (19) and

$$q_2=b_0x^2+2b_1xy+b_2y^2 \quad (b=b_0b_2-b_1^2). \tag{23}$$

They have the simultaneous invariant θ and resultant R:

$$\theta=a_0b_2-2a_1b_1+a_2b_0,\quad R=4ab-\theta^2. \tag{24}$$

Invariants (21) and (22) will now be designated Q_a and I_a. For q_2 we have invariants Q_b and I_b. We define absolute invariants K_i by

$$Q_{a+kb} = Q_a + k^\tau Q_b + \sum_{i=1}^{2\tau} k^i K_i \quad (k^{2\tau+1} \equiv k). \tag{25}$$

The values of certain K_i for various types are given in *A. J.*, §§ 8–11.

14. First, let $-a$ be a square $\neq 0$ in the field. For $a_0 \neq 0$, let r be a particular element for which $r^2 = -a$, and employ formulae (9)–(11). For $a_0 = 0$, set $r = a_1$, and employ (13). In either case, we reach type (10). Within the group G_1, a pair (10) is transformed into a similar pair only by $(\rho X, \rho^{-1} Y)$, whence r, β_0^τ, β_2^τ are unaltered, and by $(\rho Y, -\rho^{-1} X)$, whence r is changed in sign and β_0^τ, β_2^τ interchanged. In each case, $t = r(\beta_0^\tau - \beta_2^\tau)$ is unaltered. We therefore seek an invariant V_a of the general pair of forms such that $V_a = 0$ when $-a$ is zero or a not-square, and $V_a = t$ when $-a = r^2$. We thus set

$$V_a = \tfrac{1}{2}\{1 + (-a)^\tau\} W. \tag{26}$$

First, let the coefficients be such that $-a$ is the square of an element $r \neq 0$. Then $V_a = W$. For $a_0 \neq 0$, we may employ (11) and get $t = F$,

$$F = r\{(\delta - 2rd)^\tau - (\delta + 2rd)^\tau\}. \tag{27}$$

Since F involves only even powers of r, it equals a polynomial in the original coefficients. For $a_0 = 0$, we apply (13) and get $t = L$,

$$L = a_1\{b_0^\tau - (a_2^2 b_0 - 4a_1 a_2 b_1 + 4a_1^2 b_2)^\tau\}.$$

For any a_0, $W \equiv F + M(1 - a_0^{2\tau})$. Set $a_0 = 0$; then $L = F_1 + M$, where F_1 is the value $a_1 b_0^\tau$ of F for $a_0 = 0$. Hence

$$W \equiv F - a_1(a_2^2 b_0 - 4a_1 a_2 b_1 + 4a_1^2 b_2)^\tau (1 - a_0^{2\tau}). \tag{28}$$

Next, let $a = 0$. Then $F = 0$, $W^2 = 0$, since

$$a_1^2(1 - a_0^{2\tau}) = (a_1^2 - a_0 a_2)(1 - a_0^{2\tau}).$$

Hence the invariant V_a with the prescribed values is given by (26)–(28). Its weight is 1. In particular, for $p^n = 3$ and $p^n = 5$, we obtain, respectively,

$$V_a = b_1(a_1^2 + a_0 a_2)(a_2 - a_0) + a_1 b_2(a_0^2 - 1) + a_1 b_0(1 - a_2^2), \tag{29}$$

$$V_a = b_0^2 a_1(1 - a_2^4) + b_2^2 a_1(a_0^4 - 1) + (b_1^2 - 2b_0 b_2)(a_1^2 - 2a_0 a_2)a_1(a_0^2 - a_2^2) + b_1 b_2 P - b_1 b_0 P', \tag{30}$$

where

$$P = 2a_1^4 a_2(1 - a_0^4) - a_0^3 a(1 + a^2) = 2a_1^4 a_2 + 2a_0^3 a_1^2 - 2a_0 a_1^2 a_2^2 - a_0^2 a_2^3 - a_0^4 a_2, \tag{31}$$

while P' is derived from P by permuting a_0 with a_2 and changing the sign of a_1. By inspection, (29) and (30) are unaltered by $(y, -x)$.

The pair (10) may be transformed by (ρ, ρ^{-1}) into a pair with

$$\beta_0 = 1 \text{ or } \nu_1; \quad \beta_0 = 0, \quad \beta_2 = 0, 1, \nu_1 \quad (\nu_1 \text{ a fixed not-square}). \tag{32}$$

The resulting types are differentiated by the invariants

$$a = -r^2, \quad b = \beta_0\beta_2 + \tfrac{1}{4}a^{-1}\theta^2, \quad \theta, \quad K_\tau = \beta_0^\tau + \beta_2^\tau, \quad V_a = r(\beta_0^\tau - \beta_2^\tau). \tag{33}$$

15. Next, let $a = 0$, $I_a = 0$, so that q_1 is equivalent under G_1 with a_0x^2, where $a_0 = 1$ or ν_1 according to the value of the invariant $Q_a = -a_0^\tau$. Within G_1 an automorph of q_1 transforms q_2 into

$$\theta^{-1}a_0bx^2 + a_0^{-1}\theta y^2 \ (\theta \neq 0), \quad 2a_0^{-\tau}V_bxy \ (\theta = 0, b \neq 0), \quad 2a_0^{1-\tau}K_1x^2 \ (\theta = b = 0), \tag{34}$$

in which V_b is derived from V_a (§ 14) by interchanging the a's and b's.

If $I_a = 1$, so that $q_1 \equiv 0$, invariants b and Q_b characterize the types q_2.

16. Finally, let $-a$ be a not-square ν. Within the group G_1, the pair q_1, q_2 can be transformed into

$$q_1' = x^2 - \nu y^2, \quad q_2' = ex^2 + 2fxy + gy^2, \tag{35}$$

$$\nu = -a, \quad g - e\nu = \theta, \quad eg - f^2 = b. \tag{36}$$

We enlarge the $GF[p^n]$ to the $GF[p^{2n}]$ by adjoining a root of

$$\varepsilon^2 = \nu. \tag{37}$$

By the transformation of determinant unity,

$$X = (x - \varepsilon y)/(2\varepsilon), \quad Y = x + \varepsilon y, \tag{38}$$

$$x = \varepsilon X + \tfrac{1}{2}Y, \quad y = -X + \tfrac{1}{2}\varepsilon\nu^{-1}Y, \tag{38'}$$

the pair (35) becomes

$$Q_1 = 2\varepsilon XY, \quad Q_2 = \varkappa X^2 - \nu^{-1}\theta\varepsilon XY + \sigma Y^2, \tag{39}$$

$$\varkappa = k - 2f\varepsilon, \quad \sigma = (k + 2f\varepsilon)/(4\nu), \quad k = g + e\nu. \tag{40}$$

The only automorphs of determinant unity of Q_1 are

$$X' = \lambda X, \quad Y' = \lambda^{-1}Y. \tag{41}$$

The corresponding transformation T on the variables x, y has its coefficients in the $GF[p^n]$ if and only if $\lambda^{p^n} = \lambda^{-1}$, as follows from the fact that Y is the product of -2ε and the function conjugate to X with respect to the $GF[p^n]$. A direct verification follows from

$$T = \begin{pmatrix} \alpha & \nu\gamma \\ \gamma & \alpha \end{pmatrix}, \quad \alpha = \tfrac{1}{2}(\lambda^{-1} + \lambda), \quad \gamma = \tfrac{1}{2}\nu^{-1}\varepsilon(\lambda^{-1} - \lambda), \quad \varepsilon^{p^n} = -\varepsilon,$$

whence $\lambda = \alpha - \gamma\varepsilon$, $\lambda^{-1} = \alpha + \gamma\varepsilon$, $\lambda^{p^n} = \lambda^{-1}$. The equation

$$\lambda^{p^n+1} = 1 \tag{42}$$

has $p^n + 1$ solutions in the $GF[p^{2n}]$. By (40)

$$\sigma = \varkappa^{p^n}/(4\nu), \quad \varkappa^{p^n+1} = k^2 - 4\nu f^2 = \theta^2 + 4\nu b = -R. \tag{43}$$

If $R = 0$, then $\varkappa = 0$, $k = f = 0$, $q_2' = -\frac{1}{2}\nu^{-1}\theta q_1'$.

Henceforth, let $R \neq 0$. By (43), there are $p^n + 1$ pairs of forms (39). Since $\varkappa \neq 0$, (41) is an automorph of Q_2 only when $\lambda^2 = 1$. Hence the pairs of forms (39) fall into two sets each containing $\frac{1}{2}(p^n + 1)$ pairs equivalent under transformations (41), satisfying (42). *For given values of the invariants ν, b, θ, with $R \neq 0$, the pairs (35) fall into two non-equivalent sets under G_1; the sets are differentiated by the two square roots of $-R$ in the $GF[p^{2n}]$.*

To prove the last statement, set $j = \frac{1}{2}(p^n + 1)$. Under transformation (41), subject to (42), the pair (39) is replaced by a pair with $\varkappa' = \varkappa\mu$, where $\mu = \lambda^2$ is a root of $\mu^j = 1$. Hence we may restrict $\varkappa$ to two values $\varkappa_1$, $\varkappa_2$, such that $\varkappa_1^j$ and $\varkappa_2^j$ are the two square roots of $-R$.

We proceed to the construction of an invariant Ω of the general pair of forms q_1, q_2 in the $GF[p^n]$, $p > 2$, which will differentiate the two sets just mentioned. To this end, we first determine a non-vanishing of multiple of $\varkappa^j$ which is expressible rationally in terms of the coefficients of (35). Since 4 is a square and ν a not-square, $(4\nu)^j = -4\nu$. Hence, by (43),

$$\varkappa^j \pm 4\nu\sigma^j = \varkappa^j[1 \mp (-R)^\tau], \quad \tau = \tfrac{1}{2}(p^n - 1). \tag{44}$$

According as $-R$ is a not-square or a square in the $GF[p^n]$, we employ the upper or lower signs in (44) and see that the non-vanishing functions

$$U_1 = \varepsilon\{(k - 2f\varepsilon)^j - (k + 2f\varepsilon)^j\}, \quad U_2 = (k - 2f\varepsilon)^j + (k + 2f\varepsilon)^j \tag{45}$$

equal polynomials in ν, e, f, g, which are unaltered by every transformation of determinant unity which replaces (35) by a like pair. Set*

$$\Omega_1 = \tfrac{1}{2}[1-(-a)^\tau]W_1, \quad \Omega_2 = \tfrac{1}{2}[1-(-a)^\tau](-a)^{2\tau}W_2, \tag{46}$$

in which W_i is to be determined so that it shall equal U_i when the general pair of forms becomes (35). Now $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ will replace $x^2 - \nu y^2$ by (19), with $a = -\nu$, and will have determinant unity if

$$\alpha^2 - \nu\gamma^2 = a_0, \quad \beta = (a_1\alpha + \nu\gamma)/a_0, \quad \delta = (\alpha + a_1\gamma)/a_0.$$

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* Whence $\Omega_i = 0$ if $-a$ is a square $\neq 0$. For $a = 0$, $W_1 = 0$, $W_2 \neq 0$; hence in Ω_2 we insert the factor a power of $-a$.

The inverse $\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$ replaces (19) by $x^2 - \nu y^2$, and (23) by q_2' with

$$f = a_0^{-2}(sd - \alpha\gamma t), \quad k = a_0^{-2}(st - 4\nu\alpha\gamma d), \quad t \equiv a_0\theta + 2\nu b_0, \quad s \equiv \alpha^2 + \nu\gamma^2,$$

with d given by (6). Inserting the values of $f, k, \nu = -a, e = (k-\theta)/(2\nu)$ into (45), we have the required functions W_i. The resulting forms have the disadvantage that they contain the auxiliary quantities α, γ. For small values of p^n we may effect the expansions, apply $\alpha^2 - \nu\gamma^2 = a_0$, and obtain functions free of α, γ. We shall, however, make use of a different method of attack.

According as a_0 is a square h^2 or a not-square νh^2, we apply

$$S = \begin{pmatrix} h^{-1}, & -a_1 h^{-1} \\ 0, & h \end{pmatrix}, \quad N = \begin{pmatrix} a_1\nu^{-1}h^{-1}, & h^{-1} \\ -h, & 0 \end{pmatrix}$$

to the general pair of forms (19), (23). For $a_0 = h^2$, we obtain (35), with

$$e = b_0/a_0, \quad f = (a_0 b_1 - a_1 b_0)/a_0, \quad g = a_0 b_2 - 2a_1 b_1 + a_1^2 b_0/a_0. \tag{47}$$

For $a_0 = \nu h^2$, we get

$$-x^2 + \nu y^2, \quad \nu^{-1}gx^2 - 2fxy + \nu e y^2. \tag{48}$$

To the latter we apply transformation (38′) and obtain

$$-2\varepsilon XY, \quad \varkappa^{p^n}X^2 + \nu^{-1}\theta\varepsilon XY + \sigma^{p^n}Y^2, \tag{49}$$

each coefficient being conjugate with the corresponding coefficient in (39). The only transformations of determinant unity multiplying XY by -1 are

$$L = (\mu Y, -\mu^{-1}X). \tag{50}$$

By a discussion similar to that for (41), we see that L corresponds to a transformation (56) on x, y with coefficients in the $GF[p^n]$ if and only if

$$\mu^{p^n+1} = \frac{1}{4\nu}. \tag{51}$$

Applying transformation L to the pair (49), we obtain

$$2\varepsilon XY, \quad \varkappa_1 X^2 - \nu^{-1}\theta\varepsilon XY + \sigma_1 Y^2, \quad \varkappa_1 = \sigma^{p^n}/\mu^2, \quad \sigma_1 = \mu^2\varkappa^{p^n}.$$

As noted above, $(4\nu)^j = -4\nu$, so that by (43) and (51),

$$\varkappa_1^j = -\varkappa^j, \quad \sigma_1^j = -\sigma^j.$$

Hence by (44) and (45), the functions U_i for the pair (48) are the negatives of the functions U_i for the pair (35), subject to (47). Now $a_0^\tau = +$ or -1 in the respective cases. Hence in each case the desired function W_i is a_0^τ times the function obtained from U_i, in (45), by inserting the values of k and f given by (47), viz.,

$$k = \theta + 2e\nu = \theta - 2ab_0/a_0.$$

Expanding the binomials and replacing ε^2 by $-a$, we get

$$\left.\begin{aligned}\Omega_1 &= a_0^{2r-1}[(-a)^r-1]\sum_{s=1}^{(j+1)/2} C^j_{2s-1}(-a)^s(2d)^{2s-1}l^{j-2s+1},\\ \Omega_2 &= a_0^{2r-1}[(-a)^{2r}-(-a)^r]\sum_{s=0}^{j/2} C^j_{2s}(-a)^s(2d)^{2s}l^{j-2s},\\ d &= a_0b_1-a_1b_0,\quad l=a_0\theta-2ab_0.\end{aligned}\right\} \tag{52}$$

Invariant Ω_1 is of weight 3, Ω_2 of weight 2. Under $(x+ty, y)$, a_0, b_0, d are unaltered so that each Ω_i is unchanged. To complete this direct verification of the invariance of Ω_i, it remains to test the transformation $(y, -x)$. The following special forms of the Ω_i are obviously unaltered by the substitution S which interchanges a_0 and a_2, b_0 and b_2, and changes the signs of a_1, b_1. For $p^n=3$,

$$\Omega_1 = ba_0a_1a_2(a_2-a_0)+b_1(b_0-b_2)(a_0a_2+a_0^2a_2^2+a_0a_2a_1^2),$$

the product of $1+a$ by the invariant $ba_1(a_2-a_0)-ab_1(b_2-b_0)$;

$$\Omega_2 = \{b(1-a_1^2)+b_0^2+b_2^2\}(a_0^2a_2+a_0a_2^2)+a_0a_1^2a_2(a_0b_2^2+a_2b_0^2)+a_1b_1(b_0+b_2)(a_0a_2-a_0^2a_2^2).$$

For $p^n=5$, $\Omega_i = a_0a_2\Omega_i'$, where

$$\begin{aligned}\Omega_1' &= bb_1(a_1^2+2a_0a_2)(a_2^2-a_0^2)+bb_2a_1(2a_0^3+a_2m)-bb_0a_1(2a_2^3+a_0m)\\ &\qquad +b_1(b_0^3-b_2^3)(a_0a_2m-1-2a_1^4)\quad [m\equiv a_0a_2+2a_1^2],\\ \Omega_2' &= 2bb_1a_1(a_0^3+a_2^3)(1-a_0a_2a_1^2)+bb_0t+bb_2t'+b_0^3v+b_2^3v'\\ &\qquad +(b_0^2+b_2^2)b_1a_1(2a_1^2-a_0^2a_2^2a_1^2-a_0^3a_2^3),\\ t &= 2a_2^3(1-a_1^4)-2a_0^2a_2(1+a_1^4)+a_0a_1^2(1-2a_0^2a_2^2),\\ v &= a_0^3-a_0a_2^2(1-a_1^4)+a_2a_1^2(a_0^2a_2^2-2),\end{aligned}$$

t' and v' being derived from t and v by the substitution S.

The classes with $-a$ a not-square are completely characterized by the invariants a, b, θ, Ω_i. Instead of (39), we may employ canonical types with coefficients in the initial field. For the case $-R$ a not-square, let k be a fixed element for which $(k^2+R)/(4\nu)$ is a square ρ^2. Then by (43), (36_2), (40_3),

$$f=\pm\rho,\quad e=(k-\theta)/(2\nu),\quad g=(k+\theta)/2. \tag{53}$$

Hence any pair of forms with $-a$ and $-R$ not-squares is equivalent under G_1 with one of the two pairs (35), subject to (53). In view of (45_1), the sign of f is determined by the invariant Ω_1.

For $-R$ a non-vanishing square m^2, there occur types (35) with $f=0$, in view of (43). For such a type,

$$f=0,\quad e=(-\theta\mp m)/(2\nu),\quad g=(\theta\mp m)/2\quad (m=\sqrt{-R}). \tag{54}$$

If an automorph T of $x^2 - \nu y^2$ replaces $ex^2 + gy^2$ by a form lacking xy, then T is either $(\pm x, \pm y)$ or $(\nu\gamma y, \gamma x)$, where $\gamma^2\nu = -1$. Hence the two types (35), with the specification (54), are not equivalent under G_1, if -1 is a square, and may be taken as representatives of the two classes with the invariants $a = -\nu, b, \theta$. Now $\Omega_2 = 2(\mp m)^j$, so that Ω_2 determines $\mp m$, since j is here odd. But if -1 is a not-square, the two preceding types are equivalent; we then employ

$$\pm (x^2 - \nu y^2), \quad \tfrac{1}{2}(\mp \theta + m)\nu^{-1}x^2 + \tfrac{1}{2}(\pm \theta + m)y^2, \tag{55}$$

m being a fixed square root of $-R$. The two types (55) are not equivalent under G_1 when -1 is a not-square. The only transformations of determinant unity which multiply $x^2 - \nu y^2$ by -1 are

$$\begin{pmatrix} \alpha, & -\nu\gamma \\ \gamma, & -\alpha \end{pmatrix}, \quad \alpha^2 - \nu\gamma^2 = -1, \tag{56}$$

viz., transformations (50) on the original variables. If (56) replaces (55_2), by a form lacking xy, then $m\alpha\gamma = 0$, whereas $\alpha \neq 0, \gamma \neq 0$, -1 and ν being not-squares. For (55), $\Omega_2 = \pm 2m^j$. Now j is even. Hence Ω_2 differentiates the two types.

THEOREM. *As a fundamental system of invariants of a pair of binary quadratic forms in the* $GF[p^n]$, $p > 2$, *we may take**

$$a, \ b, \ \theta, \ Q_a, \ Q_b, \ K_1, \ K_\tau, \ V_a, \ V_b, \ \Omega_1, \ \Omega_2. \tag{57}$$

17. To determine a complete set of linearly independent invariants, we employ the method of § 6. The invariants in any line of the table are linearly independent, equal in number to the classes in that line, and vanish for the classes in the later lines.

CLASSES.	INVARIANTS.	NUMBER.
$-a$, $-R$ not-squares	$a^s R^\sigma \theta^i \Omega_1$, $[(-a)^\tau - 1][(-R)^\tau - 1] a^{s+1} R^{\sigma+1} \theta^i$	$2\tau^2 p^n$
$-R$ a square, $-a$ not	$a^s R^\sigma \theta^i \Omega_2$, $[(-a)^\tau - 1] a^{s+1} R^{\sigma+1} \theta^i$	$2\tau^2 p^n$
$-a$ not-square, $R = 0$	$[(-a)^\tau - 1] a^{s+1} \theta^i$	τp^n
$-a =$ square	$a^{s+1} \theta^i (b^r, b^t K_\tau, b^t V_a, K_\tau^2, K_\tau V_a)$	$\tau p^n (2p^n + 3)$
(34_1)	$b^i \theta^q Q_a^e$	$4\tau p^n$
(34_2)	$b^{s+1} Q_a^e$, $b^{s+1} Q_a^e V_b$	4τ
(34_3)	$Q_a^e K_1^i$	$2p^n$
$q_1 \equiv 0$	b^q, Q_b^e, 1	$p^n + 2$

* In place of Ω_2, we may employ $K_{2\tau}$ and K_1 (in differentiating the classes at the end of this section). Cf. *A. J.*, § 10.

The exponents take independently the following values:

$s, \sigma = 0, 1, .., \tau - 1$; $t = 0, 1, .., \tau$; $i, r = 0, 1, .., 2\tau$; $q = 1, .., 2\tau$; $e = 1, 2$.

The total number of linearly independent invariants is thus

$$2p^{3n} + p^{2n} + 2p^n.$$

Obvious linear combinations of those in the table give a simpler set.

Binary Quadratic and Linear Form in the $GF[p^n]$, $p > 2$.

18. Consider the pair of forms

$$q = a_0x^2 + 2a_1xy + a_2y^2, \quad l = rx + sy. \tag{58}$$

In addition to the invariants in § 12, we have

$$R = a_0s^2 - 2a_1sr + a_2r^2, \quad J = (1 - r^\mu)(1 - s^\mu), \tag{59}$$

where R is the resultant of the pair (58) and $\mu = p^n - 1$.

We construct an absolute invariant V with the value α when $q \equiv \alpha l^2$, $l \not\equiv 0$, and with the value zero in the remaining cases. Thus

$$V = v(1 - R^\mu)(1 - a^\mu), \quad a \equiv |q|.$$

To determine v consider the pairs (58) with $R = a = 0$; then

$$q \equiv \alpha l^2, \quad a_0 = \alpha r^2, \quad a_1 = \alpha rs, \quad a_2 = \alpha s^2. \tag{60}$$

First, take $r \neq 0$. Then $v = a_0/r^2$. Hence for any r,

$$v \equiv a_0 r^{\mu-2} + k(1 - r^\mu).$$

Next, take $r = 0$, $s \neq 0$. Then $v = a_2/s^2 = k$. Thus, for any s,

$$k = a_2 s^{\mu-2} + c(1 - s^\mu).$$

For $r = s = 0$, $v = 0$ by definition. Hence $c = 0$. Thus

$$V = (1 - R^\mu)(1 - a^\mu)\{a_0 r^{\mu-2} + a_2 s^{\mu-2}(1 - r^\mu)\}. \tag{61}$$

Finally, we shall need an invariant K with the value β when

$$q = 2\beta l l_1, \quad l_1 = \gamma x + \delta y, \quad r\delta - s\gamma = 1, \quad \beta \neq 0, \tag{62}$$

and with the value zero when q is not the product of two distinct linear factors one of which is $l \not\equiv 0$. Thus $K = \varkappa(1 - R^\mu)$. To determine $\varkappa$, consider the pairs (58) with $R = 0$, $l \not\equiv 0$, q/l^2 not a constant. Then (62) holds, whence

$$a_0 = 2\beta r\gamma, \quad a_1 = \beta(r\delta + s\gamma), \quad a_2 = 2\beta s\delta. \tag{63}$$

First, take $r \neq 0$. Then by (62_3) and (63_2), $a_1 = \beta(2s\gamma + 1)$. Eliminating γ by (63_1), we get $\beta = a_1 - a_0 s/r$. Hence, for every r,

$$x = a_1 - a_0 s r^{\mu-1} + m(1 - r^\mu).$$

Next, take $r = 0$, $s \neq 0$. Then $s\gamma = -1$, $\beta s\gamma = a_1$. Thus

$$x = \beta = -a_1, \quad x = a_1 + m.$$

Thus, for every s, $m = -2a_1 + d(1 - s^\mu)$. Finally, take $r = s = 0$, so that $x = 0$ by definition; while $x = -a_1 + d$. Hence $d = a_1$,

$$K = (1 - R^\mu)\{a_1 r^\mu - a_0 s r^{\mu-1} - a_1 s^\mu (1 - r^\mu)\}. \tag{64}$$

It remains to test the pairs for which (60) holds; but the second factor in (64) then vanishes. Hence K is an invariant of weight 1.

We may now characterize invariantively a complete set of canonical types, under the group G_1, of pairs of forms (58).

$$\begin{array}{ll} J = 0,\ R \neq 0: & x,\ R^{-1}ax^2 + Ry^2. \\ J = R = 0,\ a \neq 0: & x,\ 2Kxy. \\ J = R = a = 0: & x,\ Vx^2. \end{array}$$

For $J = 1$, then, $l \equiv 0$ and the types (20) for q are characterized by a and Q (§ 12).

Hence J, R, a, Q, K, V form a fundamental system of invariants. As a complete set of linearly independent invariants we may take

$$1,\ J,\ Q,\ Q^2,\ R^i,\ a^i,\ R^i a^j,\ K^i,\ V^i \quad (i, j = 1, \ldots, \mu). \tag{65}$$

Any product of the invariants (65) can be expressed as a linear function of them by means of the following relations:

$$J^2 = J,\ JQ = Q - V^{\mu/2} + R^{\mu/2}(a^\mu - 1),\ JR = JK = JV = 0,$$
$$Ja = a(1 - R^\mu) + K^2,\ Q^3 = Q,\ Qa = 0,\ QR = (1 - a^\mu)R^{1+\mu/2},\ QK = 0,$$
$$QV = V^{1+\mu/2},\ RK = RV = aV = KV = 0,\ aK = -K^3,\ t^{\mu+1} = t\ (t = R, a, K, V).$$

Defining the invariant I by (22), we have

$$(1 - I)(1 - J) = R^\mu + K^\mu + V^\mu.$$

THE UNIVERSITY OF CHICAGO, *January*, 1909.

*Symmetric Binary Forms and Involutions.—Continued.**

BY ARTHUR B. COBLE.

§ 10. *The Form $H_{8,2}$ as a Ternary Quartic.*

In the representation of § 2, the symmetric form

$$H_{8,2} \equiv (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2(a_4x_4)^2$$

is viewed as a ternary quartic with reference to an arbitrarily chosen proper norm-conic N. If no conditions are imposed on the form the quartic is general. An involutive set of the form determines a set of five points on N such that any four of the five are apolar to the quartic, *i. e.*, an antorthic five-point or $A^5_{4,4}$ of the quartic. The $A^5_{4,4}$'s are subject to five conditions and are ∞^5 in number. A conic will pass through each one. If N be chosen as such a conic, $H_{8,2}$ has one involutive set and therefore, according to (26), has ∞^1 involutive sets. Hence

(83) *The ∞^5 antorthic sets $A^5_{4,4}$ of a general quartic are distributed ∞^1 at a time on each of a system of ∞^4 conics. On each conic of the system the $A^5_{4,4}$'s constitute the involution $\overline{H}_{1,4}$ of* (26).

In the representation of § 3, the translation of (26) is:

(84) *If a pentahedron in S_4 has its five points on a quadric and its five S_3's on a norm-curve N_4, there are ∞^1 pentahedra with points on the quadric and S_3's on N_4.*

The next and last stage in the number of involutive sets of $H_{8,2}$ is when it has ∞^3 sets and becomes an $I_{8,2}$. Hereafter we shall suppose this to be the case. The quartic, Q, has then ∞^3 $A^5_{4,4}$'s on the norm-conic N. If a point x_4 on N be fixed, the $I_{8,2}$ reduces to an $I_{2,2}$ and N is an involution conic of a cubic, f, the polar cubic of x_4 as to N. Since the invariant, $\sqrt{\frac{1}{4}S}$, of this polar cubic is of the second degree in the coefficients of the $I_{2,2}$, there are four points on N whose polar cubics have $S=0$. These four points are determined by the equation

$$(85) \qquad (a_1a_1')(a_2a_2')[(a_1a_2)(a_1'a_2')+(a_1a_2')(a_1'a_2)](a_3a_3')^2(a_4x)^2(a_4'x)^2=0.$$

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* This article is a continuation of one under the same title in this volume, pp. 183–212. The numbering of paragraphs and theorems is consecutive with that of the previous paper.

Let the four parameters of points on N, determined by (85), be y_1, y_2, y_3, y_4. Since N is an involution conic of the polar cubic of y_4, for which $S=0$, it must pass through the vertices of the Hessian triangle of the polar cubic. But these also are known to be points on the covariant S of the quartic Q and therefore must be the points y_1, y_2, y_3. Hence the four points are a symmetrical set on N; the polar conic of any two is the square of the line joining the other two or any three of the four are apolar to the quartic. *If $H_{3,3}$ is an $I_{3,2}$, the quartic Q has an antorthic set $A^4_{3,4}$.** To show that this condition is sufficient and to find the number of conics N when it is satisfied, we employ the convenient canonical form of a quartic with an $A^4_{3,4}$, namely, a sum of the fourth powers of the six lines joining the four points. Let then the four points be the vertices of the triangle of reference and the unit point.

$$(86)\qquad Q=ax_1^4+bx_2^4+cx_3^4+l(x_2-x_3)^4+m(x_3-x_1)^4+n(x_1-x_2)^4.$$

Since the conic N must contain $\infty^3\, A^5_{4,4}$'s, the polar conic of any two points y and z on N as to Q must be (when taken in line form) apolar to N. Since also N must pass through the $A^4_{3,4}$ its equation is

$$\alpha x_1(x_2-x_3)+\beta x_2(x_3-x_1)+\gamma x_3(x_1-x_2),$$

where

$$\alpha+\beta+\gamma=0.$$

Then if y and z are two points on N, we have

$$(\beta-\gamma):(\gamma-\alpha):(\alpha-\beta)=$$
$$y_1(y_2-y_3):y_2(y_3-y_1):y_3(y_1-y_2)=z_1(z_2-z_3):z_2(z_3-z_1):z_3(z_1-z_2).$$

Forming the line equation of the polar conic of y and z as to Q and substituting for the coordinates of y and z their values in terms of α, β, γ, the apolarity condition reduces to

$$(87)\qquad al(\beta-\gamma)^3+bm(\gamma-\alpha)^3+cn(\alpha-\beta)^3=0.$$

Hence *if Q has an $A^4_{3,4}$, there are three conics on the $A^4_{3,4}$ each of which contains $\infty^3\, A^5_{4,4}$'s of Q which constitute an $I_{3,2}$.* Such conics will be called *involution conics of Q.*

The binary cubic (87) is apolar to the cubic $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ and its Hessian is

$$bcmn(\gamma-\alpha)(\alpha-\beta)+canl(\alpha-\beta)(\beta-\gamma)+ablm(\beta-\gamma)(\gamma-\alpha).$$

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* Quartics of this type have been noticed briefly by Caporali; *Memoire*, p. 347, §§ 35, 36, 37, 40 and 44.

But $\beta-\gamma=0$ determines the parameter of a degenerate conic in the pencil through $A^4_{3,4}$. Hence the two conics of the pencil which form the Hessian pair of the three involution conics are

$$bcmn\, x_2x_3(x_3-x_1)(x_1-x_2)+\ldots,$$

which according to Caporali is the covariant S. Thus the triad of involution conics is completely defined by its apolarity with the triad of line pairs on $A^4_{3,4}$ and with the pair of conics into which S degenerates.

That Q have an $A^4_{3,4}$ is a single invariant condition which is easily deduced from the canonical form (86). Denote this invariant by S_1 because of its resemblance to the invariant S of a cubic. Caporali calculates the invariant of Q of the third degree

$$A=C_{3,0,0}=(a+b+c)(mn+nl+lm)+al(b+c)$$
$$+bm(c+a)+cn(a+b)+abc.$$

The invariant B of degree six which is the condition that Q have an apolar conic is

$$B=C_{6,0,0}=abclmn.$$

The covariant conic of degree five which, when $B=0$, is the point equation of the apolar conic is

$$C=C_{5,2,0}=B\left\{\frac{x_1^2}{l}+\frac{x_2^2}{m}+\frac{x_3^2}{n}+\frac{(x_2-x_3)^2}{a}+\frac{(x_3-x_1)^2}{b}+\frac{(x_1-x_2)^2}{c}\right\}.$$

The invariant D of degree fifteen, the discriminant of C, turns out to be B^2A, *i. e.*,

$$D-B^2A^*=0.$$

Since in general D does not factor into B^2A,† this is an invariant condition satisfied by our special quartic and furnishes the required invariant, $S_1=D-B^2A$, of degree fifteen. Summing up the above we find that

(88) *The involution form, $I_{3,2}$, with reference to a norm-conic, N, represents a ternary quartic which satisfies the single invariant condition of degree fifteen, $S_1=0$. Q has an $A^4_{3,4}$ and the covariant S of Q is two conics on the $A^4_{3,4}$. In the pencil of conics on $A^4_{3,4}$ there is a set of three which is apolar to the set of three degenerate conics of the pencil and whose Hessian pair is S. On each of these three involution conics, one of which is N, the sets $A^5_{4,5}$ of Q lie in an $I_{3,2}$.*

* The numerical factors of the invariants are not adjusted to any given system.

† *E. g.*, D is irreducible for the special quartic of Salmon which contains only even powers.

The simplest picture of an $I_{8,2}$ is the system of intersections of a rational quintic R_5 in S_3 by the planes of S_3. Hence we have shown that

(89) *The rational invariant theory of the rational quintic in S_3 coincides with that of a ternary quartic subject to the condition $S_1 = 0$ after the adjunction of the cubic irrationality which separates the three involution conics.**

To every comitant of Q there corresponds a binary comitant of the form $I_{8,2}$, the transition being made with formulae (29) and (30). And to every comitant of $I_{8,2}$ there corresponds a comitant of the ternary form Q which may, however, according to (89), be irrational. Each covariant of Q cuts the involution conic N in a set of points whose parameters are the parameters of a set of covariant points on R_5; *e.g.*, Q itself determines on N the parameters of the eight hyperosculating planes on R_5. The Hessian of the polar cubic of any point x_1 on N cuts N in three pairs of corresponding points; each pair with x_1 determines the parameters of a triple secant of R_5. The covariant S determines on N the parameters of the four meets with R_5 of its unique fourfold secant.

Let us consider the 24 meets of N and the Steinerian of Q, $R = T^2 - \frac{1}{6}S^3$, a curve of order twelve. If x_1 is the parameter on N of one of the 24 points, for this fixed x_1, $I_{8,2}$ becomes an $I_{2,2}$ represented by the nodal polar cubic of x_1 for which N is an involution conic. Since the discriminant of f (for any point x_1 on N) breaks up† into two rational factors of degree six, the 24 points separate into two sets of 12 points. The involution conic passes through the nodes of the polar cubics of the points of the one set. The node and flex-line of the polar cubics of points of the other set are pole and polar with regard to N [see pp. 205–6]. Hence

(90) *Each of the three involution conics of a ternary quartic with an $A^4_{3,4}$ meets the Steinerian in the 12 points (β) which correspond to the 12 points (α) in which it meets the Hessian. It meets the Steinerian further in 12 points (γ) of whose nodal polar cubics the node and flex-line are pole and polar as to the conic.*

To the sets (α), (β), (γ) on N there correspond on R_5 sets of twelve points described as follows:

(91) *R_5 has twelve tangents at points (α) which meet the curve again at points (β).*

*In some special cases noted hereafter the part of this theorem which concerns the number of involution conics requires modification.

†This separation does not involve a degeneration of R into two curves of order 6. Similarly for any point x_1 on N the invariant S of the polar cubic of x_1 is the square of a rational expression, but the covariant S is not the square of a conic.

(92) *A triple secant of R_5 on a point r meets R_5 in points s_1 and s_2 and contains two planes which touch R_5 at points t_1 and t_2; there are 12 points r, points (γ), such that the three triple secants on r determine three pairs s_1s_2 involutorily projective to the three pairs t_1t_2.*

To every invariant of Q there corresponds an invariant of R_5. Let us examine for example the invariant B. When $S_1=0$ and Q is in the canonical form (86), clearly $B=0$ requires that one of the six coefficients, say l, be zero. $B=0$ is also the condition that Q have an apolar conic which in this case is the point pair u_2u_3, two points of $A^4_{3,4}$. The conic C is then x_1^2 and it cuts N in the apolar point pair whose parameters are, say x_1 and x_2. The polar line of the three points x_1, t_1, t_2, where t_1 and t_2 are arbitrarily chosen on N, will cut N in the points x_2 and t_3, *i. e.*, every one of the set of $A^5_{4,5}$'s which contain x_1 contains also x_2. This requires on R_5 that every plane on the point whose parameter is x_1 meet the curve again in the point whose parameter is x_2, whence these are the parameters of a double point on R_5.

(93) *The condition that R_5 have a double point is the vanishing of the invariant B of the sixth degree in the coefficients of its $I_{3,2}$. The parameters of the double point are determined from the two double roots of $(c_1x)^2(c_2x)^2=0$, where $(c_1x_1)^2(c_2x_2)^2$ is the symmetric form of degree five in the coefficients of $I_{3,2}$ which represents the conic C.*

Instead of using known properties of Q to obtain facts concerning R_5 the reverse process can be employed. It is well known that an R_5 can lie on a quadric surface, in which case all generators of one kind are fourfold secants. The $I_{3,2}$ of such an R_5 determines an interesting ternary quartic not noticed hitherto. As before, to each fourfold secant, there corresponds on N four double points of S, whence S must be the square of the conic N and its points must fall into sets of four which belong to an $I_{1,3}$. Each set of the $I_{1,3}$ is an $A^4_{3,4}$ of Q.

We shall show first that S the square of a conic is a sufficient condition for this phenomenon, the necessity of the condition being obvious. Let s_1 be any point on the conic S. The polar cubic of s_1 as to Q has a Hessian triangle whose vertices, t_1, t_2, t_3, also lie on S. Each set of three points, $s_1t_it_k$, is apolar to Q. The polar cubic of t_1 has a Hessian triangle with vertices $s_1\tau_2\tau_3$ on S. But s_1t_2 and s_1t_3 are apolar to this cubic, whence $\overline{t_2t_3}$ is the line of the Hessian triangle opposite s_1, *i. e.*, the lines $\overline{t_2t_3}$ and $\overline{\tau_2\tau_3}$ coincide. Hence the pairs t_2t_3 and $\tau_2\tau_3$ on the same line and conic coincide. Thus any three of the four points $s_1t_1t_2t_3$ are

the Hessian triangle of the polar cubic of the remaining point and the four are an $A^4_{3,4}$ of Q. Evidently there are ∞^1 such $A^4_{3,4}$'s of Q on S in an $I_{1,3}$. That S is an involution conic follows from the fact that S is on the Hessian triangle of the polar cubic of any point on S and therefore is an involution conic of the polar cubic (p. 204). But if an $H_{3,2}$ for any assigned value of one variable reduces to an $I_{2,2}$, the $H_{3,2}$ is an $I_{3,2}$. Hence

(94) *That the covariant S of a quartic Q be the square of a conic is the necessary and sufficient condition that Q have $\infty^1 A^4_{3,4}$'s which must lie in an $I_{1,3}$ on the conic. The conic also contains $\infty^3 A^5_{4,5}$'s of Q which lie in the $I_{3,2}$ of an R_5 on a quadric surface.*

Further it is clear that

(95) *An $I_{1,3}$ on a conic determines a unique quartic $\bar{Q}$ whose covariant S is the square of the conic. The sets of $I_{1,3}$ are $A^4_{3,4}$'s of $\bar{Q}$.*

The R_5 on a quadric, the $I_{1,3}$, and the quartic $\bar{Q}$ each depend on three absolute constants. The quartic Q with a unique $A^4_{3,4}$ depends on five absolute constants. It is one condition on a, b, c, l, m, n that S be the square of a conic. Writing

$$r_1 = al, \qquad r_2 = bm, \qquad r_3 = cn,$$
$$s_1 = r_1 + r_2 + r_3, \quad s_2 = r_2r_3 + r_3r_1 + r_1r_2, \quad s_3 = r_1r_2r_3,$$

this condition is $s_2^2 - 4s_1s_3 = 0$. When this is satisfied, the number of constants is four; but the reduction to the canonical form is possible in ∞^1 ways, whence the number of absolute constants is three. In treating $\bar{Q}$ we must suppose $B \neq 0$, which emphasizes that an R_5 on a proper quadric can not have a double point.

The Steinerian and Hessian of $\bar{Q}$ have remarkable properties. It will happen six times that two points of an $A^4_{3,4}$ coincide. Let the coincident pair of such an $A^4_{3,4}$ be ρ_i and the other two points be σ_i and τ_i, $i = 1, 2, \ldots, 6$. Then $\rho_i^2\sigma_i$, $\rho_i^2\tau_i$, and $\rho_i\sigma_i\tau_i$ are each apolar to $\bar{Q}$. This requires that the polar cubics of σ_i and τ_i have cusps at ρ_i with cusp tangents $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$ respectively. So all points on the line $\overline{\sigma_i\tau_i}$ have polar cubics with a node at ρ_i and the line is a part of the Steinerian. Since the Jacobian of a net of curves has a double point at a point where all the curves of a pencil in the net have a double point, the Hessian of $\bar{Q}$ (the Jacobian of the net of first polars) has a double point at ρ_i. Evidently the Steinerian, $R = T^2 - \frac{1}{6}S^3$, factors into two sextics when S is the square of a conic. The factors are $R_1 = T + \sqrt{\frac{1}{6}}SS$ and $R_2 = T - \sqrt{\frac{1}{6}}SS$. We

shall see that, to accord with our earlier convention,* the first factor must represent the six lines $\overline{\sigma_i\tau_i}$. Since $T=0$ and $R_2=0$, when $R_1=S=0$, we find that the line $\overline{\sigma_i\tau_i}$ must meet both R_2 and T three times at σ_i and three times at τ_i. Thus R_2 and T each have the line $\overline{\sigma_i\tau_i}$ for a double flex-tangent at the flex-points σ_i and τ_i. Since the cusp tangents of first polars touch the Hessian at the cusp, the tangents to the Hessian at ρ_i are $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$.

(96) *The Hessian of Q has six double points ρ_i on the conic S whose tangents cut S again in pairs of points $\sigma_i\tau_i$. The Steinerian decomposes into the six lines $\overline{\sigma_i\tau_i}$ and a sextic R_2 which has the six lines $\overline{\sigma_i\tau_i}$ for double flex-tangents which touch at σ_i and τ_i. R_2, being in one-to-one correspondence with H, has six further double points not on S. The polar cubics of points on $\overline{\sigma_i\tau_i}$ have double points at ρ_i, the double-point tangents being harmonic with $\overline{\rho_i\tau_i}$ and $\overline{\rho_i\sigma_i}$, which are the cusp tangents of the polar cubics of σ_i and τ_i respectively.*

According to (94), $N=\sqrt{S}$ is an involution conic of $\bar{Q}$. But $\bar{Q}$ has also other involution conics. From the definition of the involution conics in (88) we conclude that when S is a perfect square two of the three involution conics on an $A^4_{8,4}$ coincide with S. But the third is distinct and still has the property that taken with the conic S twice it forms a triad apolar to the three degenerate conics on the $A^4_{8,4}$. But this requires that the involution conic be apolar to S in line form.† The four points on S being given, there is a single conic of this kind and the $I_{1,8}$ of $A^4_{8,4}$'s determines a pencil of conics M which meet in a set of four points, O^4, orthic‡ to the conic S. On an involution conic M there is a single $A^4_{8,4}$ of $\bar{Q}$ and its involution is that of an R_5 which does not lie on a quadric. As before, we show that the twelve meets of M and H correspond to the twelve meets of M and R_2, whence the sextics H and R_2 are cut by ∞^1 conics in twelve pairs of corresponding points.

Through a point x on a line $\overline{\sigma_i\tau_i}$ there passes a conic M which is an involution conic of the polar cubic of x. The invariant R_1 of this cubic vanishes;

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* In the first paper the general cubic was considered with regard to a *proper* involution conic, the polar conic of a point P as to the cubic A. If, however, P were on the Hessian of A, Δ_A, its polar conic would be degenerate and the entire construction would fail. From (54), $\Delta_A=-\frac{1}{4}R_1(B-2\sqrt{\frac{1}{3}Sf})$. The condition that P lie on Δ_A is given in (52). Comparing this condition with (49) we find it to be $R_1=0$. For this case, however, a new determination of the involution conics was made (p. 205).

† A triad of conics in a pencil, $(ax)^2$, $(bx)^2$, $(cx)^2$, is apolar to the triad of degenerate conics in the pencil if $(abc)^2=0$.

‡ *I. e.*, S in lines is a sum of squares of the four points.

hence its flex-line is the polar of its node ρ_i as to M. The polar lines of ρ_i as to the pencil M meet in a point r_i, the partner of ρ_i in the quadratic Cremona involution determined by O^4. The line $\overline{r_i\rho_i}$ is touched by two conics of the pencil at r_i and ρ_i. Since the conic on ρ_i touches N at ρ_i, the line $\overline{r_i\rho_i}$ is the tangent to $N=\sqrt{S}$ at ρ_i. This is also evident from the fact that this tangent is the flex-line of the cuspidal polar cubics of σ_i and τ_i.

(97) *In addition to the special involution conic N whose square is S, Q has ∞^1 involution conics M which lie in a pencil apolar to N in lines and cut N in the $A^4_{8,4}$'s of $\bar{Q}$. The conics M each cut the Steinerian sextic R_2 and the Hessian in twelve pairs of corresponding points.*

The flex-lines of the polar cubics of points on a line $\overline{\sigma_i\tau_i}$ meet in a point r_i, each line on r_i being a flex-line of two cubics. r_i is on the tangent to N at ρ_i and the pair of points $r_i\rho_i$ is apolar to all the conics M.

We shall show further that the six Steinerian lines touch a conic. Take ρ_i, σ_i, τ_i as the reference triangle u_1, u_2, u_3 and let $x_2+x_3=0$ be the tangent to N at u_1. Then $\bar{Q}$ takes the form

$$\alpha x_1^4+\beta x_2^4+\gamma x_3^4+\delta(x_2+x_3)^4+4\lambda x_2^3x_1+4\mu x_3^3x_1.$$

$$B=\alpha\delta\lambda^2\mu^2$$

and

$$C=\alpha\lambda^2\mu^2x_1^2-\alpha\delta\gamma\lambda^2x_2^2-\alpha\delta\beta\mu^2x_3^2+2\alpha\delta\lambda\mu\, x_1(\lambda x_2+\mu x_3).$$

Thus $x_1=0$ meets C in two points apolar to u_2u_3, the two points in which it meets N. Hence this line lies on the conic locus of lines which cut N and C in harmonic pairs.

(98) *The six Steinerian lines touch a conic, the intermediate or Clebschian of C and N.*

Further study of the comitants of $\bar{Q}$ would no doubt yield results worth noting. For example the point equation of the contravariant ϕ of degree 5 and class 4 has been determined* to be

$$2(3BS-2C^2)(2HC-S^2)-(3BH-SC)^2=0.$$

Its twenty-four cusps are the meets of $3BS-2C^2=0$ and $3BH-SC=0$. From the equation of ϕ we find that it has the six double points of H as double points with the same tangents as H. Also since $3BS-2C^2$ factors, the 24 *cusps of ϕ lie 12 at a time on two conics of the pencil determined by $\sqrt{S}$ and C.* The

*Coble, AMERICAN JOURNAL, Vol. XXVIII, pp. 344–45.

locus of lines whose associate conics degenerate is in general a sextic ψ. The ψ of $\bar{Q}$ is the square of the cubic whose lines join pairs of points in the $I_{1,8}$ on N.

The above discussion suggests a classification of non-singular R_5's into three types:

(99) *Type* (*A*). R_5 is the general quintic with a single fourfold secant. Q is the general quartic with an $A^4_{3,4}$ and three involution conics (90).

Type (*B*). R_5 has a single fourfold secant. Q has ∞^1 $A^4_{3,4}$'s. The involution conic is one of a pencil and contains a single $A^4_{3,4}$. R_5 depends on four constants, Q depends on three constants and the involution conic on one.

Type (*C*). R_5 has ∞^1 fourfold secants on a quadric. The involution conic is unique. R_5 and Q each depend on three constants.

In the usual classification according to the number of fourfold secants, types (*A*) and (*B*) coincide. In this classification with reference to Q, types (*B*) and (*C*) hang together.

To characterize the R_5 of type (*B*) we observe that on the involution conic M the 12 Steinerian points whose corresponding Hessian points do not lie on M are cut out by the six Steinerian lines and separate into six pairs. The polar cubics of a pair have the same flex-line. But this line cuts M in the fixed points of the binary involution described in (73) and (92). Hence

(100) *On an R_5 of type (B) the 12 points (γ) of (92) separate into six pairs, the points of a pair determining the same binary involution on the curve.*

This is not a very marked geometrical peculiarity of the curve. A better condition should arise from the existence of the four special points of the curve which correspond to the base points of the pencil of involution conics.

When R_5 has an actual multiple point, the number of $A^4_{3,4}$'s of Q and the number of involution conics vary.

If R_5 has a double point, Q has an apolar point pair, $p_1 p_2$. The polar cubic of either point is three lines through the other. The Hessian pairs of the two sets of three lines form a quadrilateral with opposite vertices p_1, p_2; q_1, q_2; and r_1, r_2. Then both p_1, p_2, q_1, q_2 and p_1, p_2, r_1, r_2 form an $A^4_{3,4}$ of Q. On each $A^4_{3,4}$ there are three involution conics. S is the product of the four sides of the quadrilateral.

If R_5 has two double points, Q is a sum of the fourth powers of four lines. Any two of the three pairs of opposite vertices of the quadrilateral are an $A^4_{3,4}$ of Q, there being three involution conics on each $A^4_{3,4}$. Again S is the product of the four lines.

If R_5 lies on a quadric cone and has a triple point at the vertex, Q is a sum of three fourth powers. S vanishes identically. Any conic through the vertices of the three-line is an involution conic. The vertices determine the parameters of the triple point.

I will obtain finally a covariant conic of Q which becomes S when S is the square of a conic. Taking Q in the canonical form (86) and using the notation of p. 360 we find that the conic whose polar as to Q is C is

$$K = B^2 \left\{ \Sigma \frac{1}{r_1} [u_1(u_1 + u_2 + u_3) - u_2 u_3] \right\}.$$

This conic and S involve the six coefficients only in the combinations r_1, r_2, r_3. The point equation of K is

$$\begin{aligned} C_1 \equiv B^4 \Big\{ &\Sigma \frac{1}{r_1^2} [-3x_1^2 + x_2^2 + x_3^2 - 3(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \\ &+ \Sigma \frac{2}{r_2 r_3} [-x_1^2 + x_2^2 + x_3^2 - (x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \Big\}. \end{aligned}$$

The polar of K as to S is

$$\begin{aligned} C_2 \equiv B^3 \Big\{ &\Sigma \frac{1}{r_1^2} [-3x_1^2 + 2x_2^2 + 2x_3^2 - 3(x_2 - x_3)^2 + 2(x_3 - x_1)^2 + 2(x_1 - x_2)^2] \\ &+ \Sigma \frac{1}{r_2 r_3} [-4x_1^2 + x_2^2 + x_3^2 - 4(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2] \Big\}. \end{aligned}$$

Then

$$C_1 + BC_2 = 6B^2 \{ \Sigma (r_1^3 r_2^2 - r_1^3 r_3^2 - r_1 r_2^3 r_3 + r_1 r_2 r_3^3) x_1 (x_2 - x_3) \}.$$

Evidently this covariant conic of degree twenty goes through $A^4_{3,4}$. When Q has ∞^1 $A^4_{3,4}$'s, it must either vanish identically or pass through all and coincide with S. But one easily verifies that its vanishing imposes further conditions on r_1, r_2, r_3 than the one, $s_2^2 - 4s_1 s_3 = 0$, which requires that S be the square of a conic. Hence it must coincide with S.

A discussion of the relation between a ternary rational quintic and the cubic surface determined by its $I_{2,3}$ is reserved for a later article.

BALTIMORE, *April* 15, 1909.

*A Theory of Geometrical Relations.**

By Arthur Richard Schweitzer.

Introduction.

In the present paper we aim primarily to establish the result that descriptive or projective order in n-dimensions ($n = 1, 2, 3, \ldots$) may be generated p-dimensionally ($p = 1, 2, 3, \ldots, n$), so that the corresponding geometric spaces $S_{p,n}$, $\Sigma_{p,n}$ may be said to have the index (p, n), where p denotes the dimensionality of the generating relation and n is the dimensionality of the space generated.† Each of the geometric spaces is generated by two indefinables, viz., the element point and the generating relation; the other indefinables, such as that of the ordered dyad, are more broadly logical. Provided $n \overline{\gtrless} 3$, each of the descriptive spaces $S_{p,n}$ is extensible to the usual n-dimensional projective geometry by well-known methods.‡

Among the p-dimensional relations which are effective for the generation of descriptive or projective n-space, those which involve a minimum number of points, viz., $p + 1$, are especially important. We designate such a relation, which is necessarily descriptive, by $\alpha_1 R_p \alpha_2 \ldots \alpha_{p+1}$, where $p = 1, 2, 3, \ldots$. This relational proposition may be interpreted concretely for $p = 1$ by, "α_1 precedes α_2," and hence R_1 is the well-known relation of Vailati;§ for $p = 2$ by, "if a person swims from α_2 to α_3, the point α_1 is at his right"; for $p = 3$ by, "to a person stationed at α_1 'motion' along the triangle $\alpha_2 \alpha_3 \alpha_4$ in the indicated

* Read before the American Mathematical Society under the following titles: "On the Foundations of Abstract Geometry," April, 1906; "Concerning Abstract Geometrical Relations," "Systems of Axioms for Projective Geometry," September, 1906; "On the Ausdehnungslehre of Grassmann," March, 1907; "On the Relation of Right-handedness in Geometry," September, 1907.

† The interesting memoir by F. Morley, "On the Geometry Whose Element is the 3-Point of a Plane," *Trans. Am. Math. Soc.* (1904), Vol. V, p. 467, has an important relation with our development.

‡ Klein, *Mathematische Annalen*, IV, VI; Pasch, "Vorlesungen über Neuere Geometrie"; Bonola, *Giornale di Matematiche*, XXXVIII.

§ Vailati, *Rivista di Matematica*, II, pp. 71–75.

order appears clockwise,"* hence R_3 is equivalent to a relation of (absolute) right-handedness. In general, the relation R_p is p-dimensionally transitive and alternating.†

The p-dimensional systems ${}^pR_p$ generated by the relation R_p are equivalent to the systems ${}^pK_p$ generated by the (p-dimensional) linearly transitive and symmetrical relation K_p between two ordered $(p+1)$-ads,‡ $\alpha_1 \alpha_2 \ldots . \alpha_{p+1} K_p \beta_1 \beta_2 \ldots . \beta_{p+1} (p = 1, 2, 3, \ldots .)$. For $p = 3$ the relation K_p expresses sameness of sense, or relative right-handedness. An important distinction between the systems ${}^pR_p$ and ${}^pK_p$ is seen when each system is extended to n-dimensions, $n > p$. For such an extension an infinite class of relations of the type R_p is required, whereas a single relation between $(p+1)$-ads, analogous to the relation K_p, suffices in case of the extension of the system ${}^pK_p$.

For $p \gtreqless 3$ the systems ${}^pK_p$ are made "complete," for euclidian geometry by adding a certain number of axioms which permit the introduction of the real number system in a very simple manner.§ These "complete" systems are shown to provide an elegant basis for Grassmann's Ausdehnungslehre and the exposition of the latter by Peano.‖

Finally we discuss, in addition to the relations R_p and K_p, their projective analogues and other p-dimensional relations which generate n-dimensional geometry; also we consider their mutual definition. For example, we show that the well-known relation of "betweenness" $\alpha_1 \frac{I_1}{B_1} \beta_1 \beta_2$ has the p-dimensional extensions $\alpha_1 I_p \beta_1 \beta_2 \ldots . \beta_{p+1}$ and $\alpha_1 \alpha_2 \ldots . \alpha_p B_p \beta_1 \beta_2$, where the order of the elements in the terms of each relation is immaterial. The former proposition expresses concretely, "α_1 is in the interior of the $p+1$ independent points $\beta_1 \beta_2 \ldots . \beta_{p+1}$"; the latter proposition expresses, "the p independent points $\alpha_1 \alpha_2 \ldots . \alpha_p$ are between β_1, β_2."¶

* Cf. Möbius, "Der barycentrische Calcul," *Gesammelte Werke*, I, §§ 17–20.

† For definitions, see Chapter III.

‡ If the $(p+1)$-ads are identical, the order of the elements is immaterial.

§ Cf. Grassmann, "Ausdehnungslehre," 1844, *Gesammelte Werke*, I, p. 138, note 1.

‖ Peano, "Calcolo Geometrico;" *Formulaire Mathématique*, Turin (1903), pp. 277, 338; (1905), p. 188. Cf. also Schweitzer, "On the Logical Basis of Grassmann's Extensive Algebra," *Bulletin American Math. Soc.*, November, 1908.

¶ In connection with these relations we find it desirable to formulate descriptive axioms in terms of a relation $\alpha_1 \alpha_2 S \beta_1 \beta_2 \ldots . \beta_p$, which expresses that, "$\alpha_1 \alpha_2$ are on the same side of the independent points $\beta_1 \beta_2 \ldots . \beta_p$." A comparison of these "S" systems with our "R" systems is most instructive.

CHAPTER I.

Historical Remarks on the Theory of Relations.

Previous to De Morgan the subject of relations was considered almost wholly from a metaphysical standpoint. Thus De Morgan says: *

"Much has been written on relation in all its psychological aspects except the logical one, *i. e.*, the analysis of necessary laws of thought connected with the notion of relation. The logician has hitherto carefully excluded from his science the study of relation in general; he places it among those heterogeneous *categories* which turn the porch of his temple into a magazine of raw material mixed with refuse."

Again, on the inadequacy of the previous logical treatment of relations the same author says: †

"The only relations admitted into logic, down to the present time, are those which can be signified by *is* and denied by *is not*.... Accordingly, all logical relation is affirmed to be reducible to *identity*, A is A, to *non-contradiction*, Nothing both A and not-A, and to *excluded middle*, Everything either A or not-A. These three principles, it is affirmed, dictate all the forms of inference, and evolve all the canons of syllogism.... I cannot see how, alone, they are competent to the functions assigned."

Observing that one may ignore on the logical basis the metaphysical aspect of relations, De Morgan was the first to investigate their formal theory. ‡ His suggestive memoir on relations has been of marked influence on subsequent authors and contains features which have been generally recognized as fundamental. §

Although De Morgan first emphasized the formal theory of relations, it is due to Peirce ‖ to have systematically developed his ideas and to have pointed out their adaptability to a calculus. The work of Peirce has been carried to a

* *Cambridge Philosophical Transactions*, 1864, p. 331.

On the definition of a relation J. S. Mill says (James Mill, "Analysis," II, 10): "Any objects, whether physical or mental, are related .... in virtue of any complex state of consciousness into which they both enter, even if it be a no more complex state of consciousness than that of thinking them together." Compare De Morgan, *l. c.*, p. 208; Peirce, *American Journal*, III, p. 42.

† *l. c.*, p. 335.

‡ "On the Syllogism I, II, III, IV," *Camb. Phil. Trans.*, 1847–64.

§ Cf. Peirce, *American Journal*, VII, pp. 201, 202; III, p. 21, etc.

‖ *American Journal*, III.

high degree of elaboration by Schroeder.* On the plan of his discussion of relations the latter says:†

"Bei der fast unermesslichen Mannigfaltigkeit der Richtungen, nach welchen sich die Disziplin entwickelungsfähig zeigt, der Fülle ihrer Anwendungsmöglichkeiten auf die verschiedensten Gebiete—zu denen die Begriffe von 'Endlichkeit', 'Anzahl', 'Funktion' und 'Substitution' ebensowohl gehören als wie z. B. die 'menschlichen Verwandtschaftsverhältnisse'—, bei ihrer Doppelnatur als einer *Algebra* einerseits und einer Entwickelungsform der *Logik* andrerseits, nämlich ihrer *Ausgestaltung zur Logik der Beziehungen* (*und Beziehungsbegriffe*, 'Relative') *überhaupt*, scheint es unerlässlich—soll nicht die Uebersicht leiden und der Eindruck der Schönheit und Konsequenz des Ganzen verloren gehen—dass wir die verschiedenen Gesichtspunkte, unter welchen unsere Theorie zu betrachten sein wird, thunlichst scharf von einander getrennt halten. Ich werde deshalb zunächst *eine* Seite der Theorie fast ausschliesslich bevorzugen, und zwar dieselbe lediglich als eine *Algebra*, einen *Kalkul* aufbauen.... Erst wenn auf diesem Wege ein gewisser Grundstock geschaffen und ein schon recht ansehnliches Kapital von absolut feststehenden Wahrheiten—Thatsachen der Deduktion—gesichert ist, gedenke ich .... auf die Fundamente der Disziplin zurück zu kommen, um deren zuerst nur einfach hingestellte Festsetzungen dann auch heuristisch zu motiviren und aus allgemein logischen Gesichtspunkten reflektirend zu erörtern, insbesondere sie als den Zwecken eben dieser Wissenschaft, der Logik, dienstbare nachzuweisen.... Meine Bezeichnungsweisen schliessen sich sehr nahe an die von Peirce‡ in *einer* seiner Abhandlungen gebrauchten an und werden die Abweichungen späterhin gekennzeichnet und gerechtfertigt."

The genesis of a "binary Relative" is thus given by Schroeder:§

"Die Disziplin geht aus von der Betrachtung eines Denkbereiches 1^1, bestehend aus '*Elementen*' $A, B, C, \ldots,$ die als einander gegenseitig ausschliessend und von dem Nichts (0) verschieden vorausgesetzt werden. Als Inbegriff dieser Elemente wird der Bereich mittelst

$$\begin{aligned} 1^1 &= A + B + C + \ldots . \\ &= \Sigma_i\, i. \end{aligned}$$

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* "Vorlesungen über die Algebra der Logik."

† *l. c.*, III, 1 (1895), p. 1.

‡ Schroeder, *l. c.*, I, p. 710, Note 9c.

§ *Mathematische Annalen*, XLVI, p. 144.

in Gestalt von deren 'indentischer Summe' (logical aggregate) dargestellt. Doch ist sogleich zu betonen, dass diese Elemente — wie die (reellen) Zahlen oder die Punkte einer Geraden (ev. Strecke) — auch ein Kontinuum bilden dürfen. Irgend zwei Elemente i und j lassen sich — etwa unter dem Gesichtspunkt einer gewissen von i zu j bestehenden '*Beziehung*' — in bestimmter Folge zu einem *Elementepaar* (oder individuellen binären Relative) $i:j$ zusammenstellen, und bildet die Gesammtheit aller erdenklichen Elementepaare

$$1^2 = \Sigma_{ij}\, i:j$$

einen zweiten aus dem ursprünglichen abgeleiteten Denkbereich, der aus den Variationen mit Wiederholungen zur zweiten Klasse von des letzteren Elementen besteht. In diesem zweiten Bereich bewegt sich unsere ganze Disziplin, und es wird unter einer binären Relative (a oder b, c, ...) nichts anderes zu verstehen sein, als ein Inbegriff (identische Summe) von Elementenpaaren (keinen, einigen, oder allen) irgendwie hervorgehoben aus genanntem Bereiche."

That is, a class of elements gives rise to a class of ordered dyads,* and one obtains a relation, say $\alpha Q \beta$, satisfied by every ordered dyad $\alpha\beta$ of a class of dyads derived from the initial class in virtue of given *principles of selection.*†

Schroeder's treatment of relations contains many valuable contributions, such as the preceding generation of a binary relative; but, as he himself indicates, it is primarily an independent discipline, an algebra of relations. It is due to Russell‡ to have developed a theory of relations which is at once free from a complicated symbolism and, as he shows, broadly accessible to mathematics. In accomplishing this task Russell has been very materially aided by the investigations of Peano.§

* On the meaning of an ordered dyad, see Chapter II.

† Royce, *Transactions of the American Mathematical Society*, VI, p. 358, has also generated relations on the selective basis.. Cf. also Schroeder, "Algebra der Logik," Vol. II, 2 (1905), Anhang 8. It is an interesting problem to arrive at our "R" or "K" systems by means of purely selective methods on the basis of a suitable existential domain; undefined principles of selection will then take the place of our formal indefinables.

‡ See his "Principles of Mathematics."

§ Cf. A. T. Shearman, "The Development of Symbolic Logic," Chapter VI. The general logical position of our paper is formal (cf. De Morgan, Russell, *l. c.*), not selective. The reader will find it interesting to compare our paper with the following sections of "Russell": §§ 53–55, 71, 81–82 (cf. 54), 83 (last lines of p. 87), 89, 96, 98, 187–208, 222, 225, etc.

CHAPTER II.

*On the Theory of n-Dimensional Chains** ($n = 1, 2, 3, \ldots$).

In the following, as indeed throughout the present paper, we assume as logical indefinable the functional ordered dyad xy in the variables x, y. A functional ordered dyad xy has the property that xy is a proper ordered dyad when x and y are given specific values from their respective ranges. If $a \neq b$, then the ordered dyads ab, ba are distinct; also if $c \neq a$ or b, or $d \neq a$ or b, the ordered dyads ab and cd are distinct. Thus the necessary and sufficient condition that $ab = cd$ is $a = c$ and $b = d$. Also the elements of an ordered dyad are not necessarily distinct.

Of the dyads ab, ba any one is called the conjugate of the other. The dyad aa is conjugate to itself. The dyads ax, xb are said to be connected, ab is their resultant and x is their element of connection. We define now a linear permutation of the elements $a_1, a_2, \ldots, a_n$ to be the set of ordered dyads formed out of these elements with the following properties:

1. Every element appears in dyads as the associate of the remaining elements.
2. A dyad and its conjugate are not both in the set.
3. If any two connected dyads are in the set, then their resultant is also in the set.

A linear permutational set in n elements $a_1, a_2, \ldots, a_n$ is then of the following type:

$$\begin{array}{cccc}
a_1 a_2 & a_2 a_3 & a_3 a_4 \ldots a_{n-1} a_n \\
 & a_1 a_3 & a_2 a_4 \ldots a_{n-2} a_n \\
 & & a_1 a_4 \ldots a_{n-3} a_n \\
 & & \ldots\ldots\ldots \quad \ldots \\
 & & \qquad\quad a_1 a_n
\end{array}$$

Two permutations are identical if they contain the same elements and the dyads of one set can be identified with the dyads of the other set.

A set of elements S gives rise to a set of dyads called a linear chain under the following conditions:

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* On the term "chain," as used in this chapter, compare article on Kinematics in *Encyclopaedie der Mathematischen Wissenschaften*, IV, 3, § 28.

1. Every element appears in at least one dyad of the set.
2. A dyad and its conjugate are not both in the set.
3. Every dyad is connected with some other dyad.
4. If there exists a dyad connected in only one element, there exists at most one other such dyad.
5. No element appears in more than two dyads of the set.

These conditions are satisfied by the following sets of dyads:

1) *ab* *bc* *ca* *ef* *fg*,
2) *ab* *bc* *ca* *ef* *fg* *ge*.

The conditions are also necessary, as is shown by these systems:

1. This condition is obviously independent.
2. *ab* *bc* *cd* *ef* *fe*.
3. *ab* *bc* *ca* *ef*.
4. *ab* *bc* *cd* *ef* *fg*.
5. *ab* *bc* *ca* *ce*.

In order that the chain of dyads satisfying 1–5 be unique it is necessary to add:*

6. If a set of dyads satisfies 1–5 it also satisfies the condition: The set of elements S is contained by any set T of elements with the following properties:

1) If there is a dyad ab such that a is not an element of connection, then T contains an element a; if there is no such dyad, then T contains some element of S.

2) T contains every element x of S such that some element u common to S and T forms with x a dyad ux in the set of dyads.

Under postulates 1–6 two cases may arise: there exists a dyad which is connected in only one element, or there is no such dyad. In the former case the chain is said to be open; in the latter, it is closed. An open chain in a finite number of elements contains two dyads, ab, kl, such that a and l are not elements of connection; a and l are called the first and last elements of the chain respectively. There is, moreover, a unique second element in an open chain, a unique third, etc. An important property is the following: Every linear permutation contains uniquely an open linear chain in the same number of elements. Conversely, given an open linear chain of dyads in a finite number of elements we may arrive at a permutational set of dyads by adding to the set of dyads of the

* Cf. Russell, "The Principles of Mathematics," §§ 189, 229.

chain the resultants of the connected dyads in the chain, adding the resultants of the connected dyads contained in the set thus obtained, and so on.

In an interesting memoir on tactical systems E. H. Moore* has given a valuable generalization of a closed linear chain by means of his systems $S\{k, l, m\}$. In fact, every closed linear chain is a system $S\{2, 1, m\}$, but not necessarily conversely; since, for example, the dyads 12, 21, 34, 43 are an $S\{2, 1, 4\}$, and the dyads 12, 23, 31, 45, 56, 64 are an $S\{2, 1, 6\}$, but contradict postulates 2 and 6.

An important generalization of open and closed linear chains may be made in the following manner. By a planar triad $[abc]$ we mean the dyads cb, ac, ba. The triad $[abc]$ has a unique conjugate, namely, $[bac]$. The three-dimensional tetrad $[abcd]$ is the set of planar triads $[dbc]$, $[adc]$, $[abd]$, $[bac]$; and $[bacd]$ is the conjugate of $[abcd]$. In general, we define the $(p-1)$-dimensional p-ad $[a_1 a_2 \ldots . a_p]$ to be the set of $(p-2)$-dimensional $(p-1)$-ads $[a_p a_2 \ldots . a_{p-1}]$ $[a_1 a_p \ldots . a_{p-1}] \ldots . [a_1 a_2 \ldots . a_{p-2} a_p]$ $[a_2 a_1 \ldots . a_{p-1}]$; and $[a_2 a_1 \ldots . a_{p-1} a_p]$ is the conjugate of $[a_1 a_2 \ldots . a_p]$. The $(p-1)$-dimensional p-ads $[x a_2 \ldots . a_p]$ $[a_1 x \ldots . a_p] \ldots . [a_1 a_2 \ldots . x]$ are connected, x is the element of connection and $[a_1 a_2 \ldots . a_p]$ is their resultant. A set of connected $(p-1)$-dimensional p-ads $(p = 2, 3, \ldots .)$ may be exhibited in the following formal way:

$$\begin{array}{ccccccccc}
 & & & [ax] & & [xb] & & & \\
 & & [axb] & & [bxc] & & [cxa] & & \\
 & [acxb] & & [bxdc] & & [cdax] & & [xabd] & \\
[bcxde] & & [edxba] & & [abxec] & & [cexad] & & [daxcb] \\
 & & & & \text{etc., etc.} & & & &
\end{array}$$

If p is odd each set is obtained from a definite set of closed linear chains and the element x by a simple process of formal interpolation; if p is even, each set is obtained from a definite set of open linear chains by a similar interpolation.† To construct now p-dimensional open chains we proceed thus. We illustrate the construction for $p = 2$, although the method is easily recognized to be general. With every planar triad $[abc]$ three systems of connected triads can be constructed in the elements x, a, b, c, each system containing the triad $[abc]$. For we need only find the sets of connected triads of which each of the triads

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* *American Journal*, Vol. XVIII, p. 268.

† The p-dimensional open and closed chains which we shall construct below are made accessible to the substitution theory by exhibiting them formally in a manner analogous to the above exhibition of connected p-ads.

$[xbc]$, $[axc]$, $[abx]$ is the resultant respectively; we get then

$$\begin{matrix} [abc] & [xac] & [xba] \\ [bxc] & [abc] & [axb] \\ [cxb] & [acx] & [abc] \end{matrix}$$

Consider now the triad $[abc]$ and let $x \neq a, b, c$. This triad gives rise to an open chain of triads if 1) $[abc]$ is replaced by the three triads $[xbc]$, $[axc]$, $[abx]$, of which $[abc]$ is the resultant, 2) or to the triad $[abc]$ is added any one of the three sets of two triads which form with $[abc]$ a set of connected triads. Thus an open planar chain in four elements a, b, c, x consists of three connected triads and the triad $[abc]$ gives rise to four distinct chains in four elements. Likewise the conjugate triad $[bac]$ will give rise to four distinct chains, each of which is distinct from the former, so that the total number of planar chains in four elements is $2 . 4 = 8$. Each of the preceding chains may be lengthened in a manner analogous to the above, viz., we replace any triad of a chain by the three triads of which it is the resultant or we add to the given chain of triads any one of the three sets of two triads which are connected with the boundary of the chain of triads. The boundary of a chain of triads is that triad which gives rise to the given chain under the following successive processes:

1. Replacing the triad by a set of triads of which it is the resultant;
2. Replacing some one of the set of triads thus obtained by a corresponding set of connected triads; and so on.

The boundary of a set of connected triads is evidently their resultant.

It is easily seen that five elements give rise to $2 . 4 . 6 = 48$ planar chains and that the total number of planar chains in n elements $(n = 3, 4, \ldots .)$ is

$$2 . 4 . 6 \ldots . 2(n-2) = 2^{n-2} . (n-2)!$$

For three-dimensional chains the discussion analogous to that for planar chains may now be carried out. We find that the complete number of spatial chains in n-elements $(n = 4, 5, 6, \ldots .)$ is $2 . 5 . 8 \ldots . (3n - 10)$. The preceding is also easily extended to p-dimensions $p \gtrless 1$. We may show that the complete number of p-dimensional chains in n-elements $1 \gtrless p < n$ is

$$N = (0 . p + 2)(1 . p + 2)(2p + 2) \ldots . ([n - p - 1] p + 2).$$

Thus, for $p = 1$, $N = n!$; $p = 2$, $N = 2^{n-2}(n-2)!$; etc.

The preceding p-dimensional chains were open; if to each such chain we add the conjugate of its boundary, we get closed chains.

From a p-dimensional chain arises a p-dimensional permutation by adding to the $(p+1)$-ads of the chain the resultants of the connected $(p+1)$-ads in the chain, adding the resultants of the connected $(p+1)$-ads contained in the set thus obtained, and so on. Of course, a p-dimensional permutation may be formally defined independently of a p-dimensional chain, and vice versa. Evidently a p-dimensional permutation in n elements contains a p-dimensional chain in the same number of elements. The formal theory of p-dimensional permutations is, moreover, intimately related* with the p-dimensional generation of p-space, which is discussed in Chapter IV.

CHAPTER III.

On the General Theory of Relations, Classes, and Operations.

As formal types of relational, class, and operational propositions we take aRb, $a\varepsilon C(b)$, $O_b(a)$ respectively. These propositions may be considered as expressing, "a possesses the relation R with reference to b," "a is in the class C with reference to b," "the operation O with reference to b affects a." With respect to the interdependence of the preceding propositions, we assume that

$$aRb \sim a\varepsilon C(b) \sim O_b(a);$$

that is, they are equivalent to one another.†

We assume the following analysis of the above propositions:

$$\begin{aligned} aRb &= aR + b = a + Rb, \\ a\varepsilon C(b) &= aC + b = a + Cb, \\ O_b(a) &= Oa + b = a + bO. \end{aligned}$$

The symbol of composition "+" is an indefinable which implies no ordering of the terms. The symbols aR, aC, Oa may be interpreted concretely by, "a possesses the relation R," "a is in the class C," "the operation O affects a." The symbols Rb, Cb, bO may be interpreted by, "the relation R refers to b," "the class C refers to b," "the operation O refers to b." We call aR and Ra the regressive and progressive relational associates of a respectively; etc. Two such associates are always distinct; and two regressive (progressive) associates

* For example, consider the planar permutation in 5 elements:

[423] [543] [153] [145] [124] [143] [123].

In the plane we have then the transitive property that [423] [143] [124] and [543] [153] [145] imply [523] [153] [125].

† Professor George H. Mead suggests that it may be better to assume aRb *implies* $a\varepsilon C(b)$ *implies* $O_b(a)$.

bR, cR (Rb, Rc) are identical only if their terms b, c are identical. Further, if aRb, then for any relational associate Rc, a is distinct from Rc. Thus if aRb and cRd are identical, we must have $a=c$, $Rb=Rd$; similarly, $b=d$, $aR=cR$; that is, $a=c$ and $b=d$. Hence if $a \neq b$, aRb and bRa are distinct. The terms of a relational proposition have, therefore, the character of an ordered dyad.*

We consider the equivalence

$$aRb \sim a \varepsilon C(b) \sim O_b(a).$$

Let $a'b'$ and ab be distinct dyads and suppose that for one of the members of the above equivalence, say aRb, we have $a'Rb'$; then we assume that

$$a'Rb' \sim a' \varepsilon C(b') \sim O_{b'}(a').$$

Let us assume also that

$$aRb \sim abR_1 ab.$$

Then ab is an ordered dyad. Finally, if $aRb \sim abR_1 ab$, and $a'b' \neq ab$, we assume that

$$a'Rb' \sim a'b'R_1 a'b'.$$

Thus from the preceding it follows that for any two terms x, y such that xRy, the latter proposition is equivalent to $xyR_1 xy$ and therefore to $xy \varepsilon C(xy)$. That is, every relation can be interpreted in terms of a class of ordered dyads.

By the ordered n-ad $a_1 a_2 \ldots a_n$, $n \geqq 2$ we mean some one of the sets of ordered dyads obtained from the open linear chain $x_1 x_2 \; x_2 x_3 \ldots x_{n-1} x_n$ of dyads in n distinct variables. Thus the elements of an ordered n-ad are not necessarily distinct.

Now the terms of the proposition aRb may be, in particular, an n-ad and a p-ad. If the n-ad and the p-ad are identical the relational proposition is of the type $a_1 a_2 \ldots a_n R a_1 a_2 \ldots a_n$, which we write $a_1 a_2 \ldots a_n R_{12 \ldots n}$, the subscript of the R being a linear open chain. In this case we say that the $a_1, a_2, \ldots, a_n$ are mutually related. We postulate that from the preceding proposition $n!$ derived (and equivalent) propositions exist, viz.,

$$\{ a_{j_1} a_{j_2} \ldots a_{j_n} R_{j_1 j_2 \ldots j_n} \},$$

including the identity, *i. e.*, the given proposition. The relations

$$\{ R_{j_1 j_2 \ldots j_n} \}$$

are called the derived relations of the relation $R_{12 \ldots n}$. In particular, if $n=2$, R_{21} is the conjugate of the relation R_{12}. $R_{j_1 j_2 \ldots j_n}$ is an even derived relation

* We need hardly say that the analysis in this paragraph and the equivalence in the preceding paragraph are speculative. The reader may compare our remarks with Russell, *l. c.*, §§ 38, 71, 76, 77, 96, 212–215.

of $R_{12\ldots n}$ if the $(n-1)$-dimensional n-ads (cf. Chapter II) $[12\ldots n]$ and $[j_1 j_2 \ldots j_n]$ are identical; otherwise $R_{j_1 j_2 \ldots j_n}$ is an odd derived relation of $R_{12\ldots n}$. If all the derived relations of a given relation are identical, then the given relation is symmetrical; if only the even derived relations are identical, the given relation is alternating.*

Mutual relations may or may not be transitive. Transitiveness is either linear, or planar, ... or n-dimensional. We define the mutual relation R to be n-dimensionally transitive $(n = 1, 2, 3, \ldots)$ if the propositions (dropping for convenience the subscript of the R)

$$\begin{array}{l} x\, a_2 \ldots a_{n+1}\, R, \\ a_1\, x \ldots a_{n+1}\, R, \\ \ldots\ldots\ldots\ldots, \\ a_1\, a_2 \ldots\ldots x\, R \end{array}$$

imply

$$a_1\, a_2 \ldots a_{n+1}\, R.$$

The analogous definition in the case of relations which are not mutual is easily found.

If $a_1 a_2 \ldots a_n R_{12\ldots n}$ is a mutual relation, we assume that it is equivalent to the mutual relation

$$b_1\, b_2 \ldots b_k\, Q_{12\ldots k}^{j'_1 j'_2 \ldots j'_k},$$

where

$$\begin{array}{lll} b_1 = a_1\, a_2 \ldots a_{j_1} & j'_1 = j_1, & \\ b_2 = a_{j_1+1}\, a_{j_1+2} \ldots a_{j_2} & j'_2 = j_2 - j_1, & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & & \\ b_k = a_{j_{k-1}+1}\, a_{j_{k-1}+2} \ldots a_{j_k} & j'_k = j_k - j_{k-1}, & (j_k = n) \end{array}$$

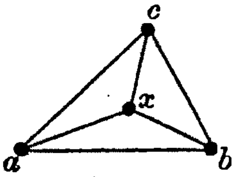
the numbers $j_1 j_2 \ldots j_{k-1}$ being chosen arbitrarily. It is also assumed that the mutual relation $a_1 a_2 \ldots a_n R_{12\ldots n}$ is equivalent to each of the following relations:

$$\begin{array}{l} a_1\, R^{(1)}\, a_2 \ldots a_n, \\ a_1\, a_2\, R^{(2)}\, a_3 \ldots a_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ a_1\, a_2 \ldots a_{n-1}\, R^{(n-1)}\, a_n. \end{array}$$

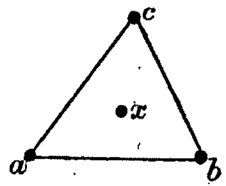
An illustration of the latter equivalence is easily given. Consider a point x in the interior of the triangle abc. We may look upon these four points as

* For the suggestion to use this term in the present connection we are indebted to Professor E. H. Moore. If the relation $R_{123\ldots n}$ is alternating, then the alternating relation $R_{213\ldots n}$ is called the conjugate of $R_{12\ldots n}$, and conversely. Such a conjugate will sometimes be denoted by $\breve{R}$.

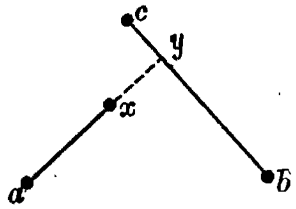
constituting a system; the points are then mutually related, *i. e.*, we have $xabc\,I\,xabc$. Then the statement that x is in the interior of abc expresses



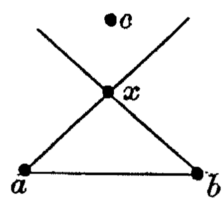
$xabc\,I\,xabc$



$x\,I^{(1)}\,abc$



$xa\,I^{(2)}\,bc$



$xab\,I^{(3)}\,c$

$x I^{(1)} abc$. $xa I^{(2)} bc$ is implied by the existence of y such that y is in the interior of bc and x is in the interior of ay. Finally, $xab\,I^{(3)}c$ is expressed by the statement that the compartment of the triangle xab at the vertex x has c in its interior.

CHAPTER IV.*

Geometrical Relations: The Systems ${}^nR_n$ $(n = 1, 2, 3, \ldots.)$.

In the present chapter we give systems of axioms for descriptive geometry in terms of the relation $R_n\,(n = 1, 2, 3, \ldots.)$. For $n \gtreqless 3$ each of the corresponding systems is sufficient for projective geometry if an axiom of continuity is added; the discussion of the systems thus amplified is, however, reserved for a later chapter. Under the respective axioms, the relation R_n involves $n+1$ independent points and therefore it may be termed n-dimensional. Since the generating relation R_n is n-dimensional and involves precisely $n+1$ points, we call it fundamental. The relation R_n is, moreover, n-dimensionally transitive and alternating (cf. Chapter III).

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* In the original exposition of the system ${}^3R_3$ the author used the expression $\alpha\beta\gamma\delta = \Omega'$ to denote $\alpha R\beta\gamma\delta$. The expression $\alpha\beta\gamma\delta = \Omega'$ may be read "the tetrad $\alpha\beta\gamma\delta$ is in the class Ω'." Further, $\alpha\beta\gamma\delta = \Omega''$ was defined to be $\beta\alpha\gamma\delta = \Omega'$, and by $\alpha\beta\gamma\delta = \Omega$ was denoted, $\alpha\beta\gamma\delta = \Omega'$ or Ω''. If $\alpha\beta\gamma\delta = \Omega'$ and $\alpha'\beta'\gamma'\delta' = \Omega'$, then $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$, which may be read "$\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are in the same class," or, "the points $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ are in the same order"; similarly, if $\alpha\beta\gamma\delta = \Omega''$ and $\alpha'\beta'\gamma'\delta' = \Omega''$, then $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$. If $\alpha\beta\gamma\delta = \Omega'$ and $\alpha'\beta'\gamma'\delta' = \Omega''$, then $\alpha\beta\gamma\delta \neq \alpha'\beta'\gamma'\delta'$; and similarly if $\alpha\beta\gamma\delta = \Omega''$ and $\alpha'\beta'\gamma'\delta' = \Omega'$, $\alpha\beta\gamma\delta \neq \alpha'\beta'\gamma'\delta'$; *i. e.*, "the tetrads $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are in opposite classes" or "the points $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ are in different orders." The expression $\alpha\beta\gamma\delta = \Omega'$ may be read "the points $\alpha, \beta, \gamma, \delta$ are in the order $\alpha\beta\gamma\delta$." With reference to the term "order," as here used, compare O. Veblen, *Transactions Am. Math. Soc.*, July, 1904. For the suggestion to employ, instead of the symbols "$=$", "$\neq$", *uniformly* a relation K, the author is indebted to Professor E. H. Moore. The author takes pleasure, in this connection, in acknowledging the stimulation of Professor Moore's suggestion in the preparation of the present paper.

The systems ${}^nR_n$ will retain validity if for the generating relation $\alpha_1 R_n \alpha_2 \ldots \alpha_{n+1}$ is substituted the well-known outer (alternating) product of $n+1$ points $[\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{n+1}]$ due to Grassmann.* In this way the systems become a fundamental part of Grassmann's calculus.

System ${}^1R_1$. I. *Axioms.*

1. There exists α.†

2. The existence of α implies the existence of α_0, β_0 such that $\alpha_0 R \beta_0$ or $\beta_0 R \alpha_0$.

3. $\alpha R \beta$ implies $\beta \bar{R} \alpha$.‡

4. $\alpha R \beta$ and $\xi \neq \alpha, \beta$ imply $\alpha R \xi$ or $\xi R \beta$.

5. $\alpha R \beta$ and $\alpha R \xi$ and $\xi \neq \beta$ imply $\xi R \beta$ or $\beta R \xi$.

6. $\alpha R \beta$ implies the existence of ξ_1 such that $\beta R \xi_1$.

7. $\alpha R \beta$ implies the existence of ξ_2 such that $\xi_2 R \alpha$.

8. $\alpha R \beta$ implies the existence of ξ such that $\alpha R \xi$ and $\xi R \beta$.

II. *Definitions.*

1. ξ is on $\alpha\beta$ means, $\alpha R \beta$ and ($\alpha R \xi$ or $\xi R \beta$).

2. ξ is in the interior of $\alpha\beta$ means, $\alpha R \beta$ and $\alpha R \xi$ and $\xi R \beta$.

That is, we obtain one definition from the other by the proper interchange of "or" and "and".

III. *Theorems.*

1. $\alpha \neq \beta$ implies $\alpha R \beta$ or $\beta R \alpha$.

By axioms 1 and 2, $\alpha_0 R \beta_0$ or $\beta_0 R \alpha_0$. Suppose $\alpha_0 R \beta_0$. Then if the points α, β are the points α_0, β_0, the theorem is true. We distinguish, therefore, these cases:

1) $\alpha \neq \alpha_0, \beta_0$; $\beta \neq \alpha_0, \beta_0$. Since $\alpha_0 R \beta_0$ and $\alpha \neq \alpha_0, \beta_0$, by axiom 4, we have $\alpha R \beta_0$ or $\alpha_0 R \alpha$. If $\alpha R \beta_0$, since $\beta \neq \alpha, \beta_0$, then $\beta R \beta_0$ or $\alpha R \beta$. If $\alpha R \beta_0$ and $\beta R \beta_0$, then $\alpha R \beta$ or $\beta R \alpha$, by theorem 2. If $\alpha_0 R \alpha$, since $\beta \neq \alpha_0, \alpha$, we have $\beta R \alpha$ or $\alpha_0 R \beta$. If $\alpha_0 R \beta$, then by axiom 5, $\alpha_0 R \alpha$ and $\alpha_0 R \beta$ imply $\alpha R \beta$ or $\beta R \alpha$.

2) $\alpha = \alpha_0$ or β_0, $\beta \neq \alpha_0, \beta_0$. That is, we have $\alpha R \beta_0$ or $\alpha_0 R \alpha$. Proof as under 1).

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* Cf. *Gesammelte Werke*, "Ausdehnungslehre," 1844 and 1862. For further details of the development from this view-point, see Chapter VI.

† Greek letters are used to denote points unless otherwise specified.

‡ The rule over the R is a symbol of negation.

3) $\beta = \alpha_0$ or β_0, $\alpha \neq \alpha_0, \beta_0$. That is, $\beta R \beta_0$ or $\alpha_0 R \beta$. Proof analogous to that under 1).

2. $\alpha R \beta$ and $\xi R \beta$ and $\xi \neq \alpha$ imply $\alpha R \xi$ or $\xi R \alpha$. By axiom 7 there is a point γ such that $\gamma R \xi$. Then assuming $\alpha \neq \gamma$, we get by axiom 4, $\gamma R \alpha$ or $\alpha R \xi$. If $\gamma R \alpha$, then since $\gamma R \xi$ we have by axiom 5, $\xi R \alpha$ or $\alpha R \xi$.

3. $\alpha R \xi$ and $\xi R \beta$ imply $\alpha R \beta$. By axiom 3, $\xi \neq \alpha, \beta$, $\beta \neq \alpha$. Then since $\alpha R \xi$ and $\beta \neq \alpha, \xi$, by axiom 4, $\beta R \xi$ or $\alpha R \beta$. Since $\xi R \beta$, by axiom 3 $\beta R \xi$ is impossible and $\alpha R \beta$ is true.

We have thus shown that our linear system ${}^1R_1$ implies the system due to Vailati.* Our system has, however, advantages which arise from the particular form in which it is stated. Namely, let us inquire whether the system ${}^1R_1$ is extensible to higher dimensions or not. That is, if we omit axiom 4 from the system ${}^1R_1$ and leave the remaining axioms in force, is it possible to add axioms in terms of the relation R such that a geometry of dimensionality greater than unity results, and what is the character of this geometry? In answering this question, we observe first that there exists a finite system of elements which satisfies all the preceding axioms except axiom 4; indeed, it is the following system† of 21 dyads in 7 elements:

12	23	34	45	56	67	71
25	57	74	41	13	36	62
51	16	64	42	27	73	35

Thus the dyads form three linear closed chains; the dyads of any two closed chains can be obtained from the third by means of the substitution

$$(4) \quad (125) \quad (376).$$

Every element is common to six dyads, and on every dyad lie five elements. The dyads may be arranged into a system of 14 (planar) triads:

[712]	[625]	[351]
[234]	[574]	[164]
[456]	[413]	[427]
[671]	[362]	[735]
[125]	[367]	

* Vailati, *Rivista di Matematica*, Vol. II, pp. 71–75.

† The extension of this system to n dimensions ($n > 1$) we shall discuss elsewhere. The above system is one of an infinitude of analogous systems. Cf. *Bulletin Am. Math. Soc.*, March, 1908, p. 265.

and into the following system of 21 (linear) triads:

$\overline{162}$	$\overline{273}$	$\overline{364}$	$\overline{425}$	$\overline{516}$	$\overline{627}$	$\overline{741}$
$\overline{235}$	$\overline{567}$	$\overline{734}$	$\overline{451}$	$\overline{123}$	$\overline{356}$	$\overline{642}$
$\overline{571}$	$\overline{136}$	$\overline{674}$	$\overline{412}$	$\overline{257}$	$\overline{713}$	$\overline{345}$

It will be observed that each dyad occupies all possible positions in the triads once and only once, thus occuring in three triads. Hence the preceding system is a subset of Moore's 3-idic system $S\{3, 2, 7\}$.* It is easily verified that under the preceding system of dyads all the axioms of the system ${}^1R_1$ are effective, *i. e.*, their hypotheses are fulfilled; and that axiom 4 is contradicted, the remaining axioms being satisfied. The geometry which is represented by the preceding system of dyads is not, however, a descriptive system; consider, for example, the triangle 125: we have 3 on 12 and 25. Let us require, then, that the geometry represented by the extended system of axioms be descriptive. We define, as above:

ξ is on $\alpha\beta$ means, $\alpha R\beta$ and $(\alpha R\xi$ or $\xi R\beta)$,† and hence ξ not on $\alpha\beta$ means, $\alpha R\beta$ and $\alpha\bar{R}\xi$, $\xi\bar{R}\beta$.

Let us see in what particular form we must put this definition. Suppose that ξ is not on $\alpha\beta$ and that

$$\alpha R\beta, \qquad \alpha R_1\xi, \qquad \xi R_2\beta.$$

Then since our geometry is to be descriptive and we have ξ not on $\alpha\beta$, we must have, also, β not on $\alpha\xi$, α not on $\xi\beta$; that is,

$$\beta\bar{R}_1\xi, \qquad \alpha\bar{R}_1\beta,$$
$$\alpha\bar{R}_2\beta, \qquad \xi\bar{R}_2\alpha.$$

Different cases evidently arise according as R, R_1, R_2, $\breve{R}$, $\breve{R}_1$, $\breve{R}_2$ are distinct or not; these are in essence as follows:

1) R_1 or $R_2 = \breve{R}$; *i. e.*, $\xi R\alpha$ or $\beta R\xi$. If $\alpha R\beta$ and $\xi R\alpha$, then let $\eta' R\alpha$ and $\eta' R\beta$; hence η' is on $\xi\alpha$ and $\alpha\beta$, which is impossible since $\eta' \neq \alpha$. If $\alpha R\beta$ and $\beta R\xi$, let $\beta R\eta''$ and $\alpha R\eta''$. Then η'' is on $\alpha\beta$ and $\beta\xi$ and $\eta'' \neq \beta$. This is impossible.

* Cf. E. H. Moore, "Tactical Memoranda," *American Journal*, Vol. XVIII, pp. 268, 270. The preceding system of fourteen planar triads may be looked upon as two *triple systems* in seven elements. This connection was kindly pointed out to us by Professor Moore.

† *I. e.*, for *some* relation R.

2) $R_1 R_2 \neq R, \breve{R}$. $R_1 = R_2$ or $\breve{R}_2$. If $R_1 = R_2$, $\alpha R_1 \xi$ and $\xi R_1 \beta$. Let $\alpha R_1 \eta$ and $\xi R_1 \eta$; then η is on $\alpha \xi$, $\xi \beta$ and $\eta \neq \xi$, which is impossible. If $R_1 = \breve{R}_2$, then we have $\alpha R_1 \xi$ and $\beta R_1 \xi$; that is, β is on $\alpha \xi$.

Other cases differ from the preceding only formally, so that we may say, if three points α, β, ξ form a descriptive triangle, then the corresponding relations R, R_1, R_2, $\breve{R}$, $\breve{R}_1$, $\breve{R}_2$ are distinct from each other. Further, on the basis of the preceding we may say that if two descriptive lines intersect, then the corresponding relations

$$R_1, \quad R_2, \quad \breve{R}_1, \quad \breve{R}_2$$

are distinct. To construct, therefore, an n-dimensional descriptive geometry ($n > 1$) an infinitude of relations of the type R is required.* However, all the axioms of system ${}^1R_1$ except axiom 4 are satisfied if we define $\alpha R \beta$ and $\alpha' R \beta'$ to mean that $\alpha\beta$ and $\alpha'\beta'$ are two euclidian parallel and similarly directed† segments. Thus any set of euclidian, similarly directed parallels in n-space ($n > 1$) will satisfy all the axioms of the system ${}^1R_1$ except axiom 4; so that, on the basis of the remaining axioms, the single relation R is capable of generating an (unlimited) linear vector.

The independence of the axioms of the system ${}^1R_1$ may be briefly discussed. Axiom 1 is obviously independent, since if there is no point, the remaining axioms are not effective. For axiom 2, take one point α such that $\alpha \bar{R} \alpha$. For axiom 3, take one point α such that $\alpha R \alpha$. The independence of axiom 4 is established by the above finite system of 21 dyads; the latter system also shows that axiom 4 is necessary to prove the existence of an infinitude of points. To show that axiom 5 is independent, take as points the ordinary system of rational numbers, and the imaginary unit i ordered thus: $\alpha R \beta$ means $\alpha < \beta$, and we define $i R \beta$ if $\beta > 0$ and $\alpha R i$ if $\alpha < 0$; also $0 \bar{R} i$, $i \bar{R} 0$. Then axiom 5 is contradicted, since $(-1) R 0$ and $(-1) R i$ do not imply $i R 0$ or $0 R i$; the remaining axioms are satisfied. The independence systems for axioms 6 and 7 are the sets of positive and negative rational numbers, respectively; the independence system for axiom 8 consists of the positive and negative integers ordered in the usual manner.

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* In the extended system, instead of axiom 4, we have the axiom of transitiveness: $\alpha R \xi$ and $\xi R \beta$ imply $\alpha R \beta$.

† Not to be confounded with sameness of sense. This confusion has occurred with many authors. Similarity of direction is excluded by means of the axiom, "$\alpha R \beta$ and $\gamma R \delta$ imply $\alpha R \gamma$ or $\gamma R \alpha$"; cf. our paper, *Trans. Am. Math. Soc.*, July (1909), p. 309.

System $^2R_2$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of $\alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 R \beta_0 \gamma_0$ or $\alpha_0 R \gamma_0 \beta_0$.

3. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.

4. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.

5. $\alpha R \beta \gamma$ and $\xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.

6. $\alpha R \beta \gamma$ and $\xi R \beta \gamma$ and $\xi \neq \alpha$ imply $\xi R \gamma \alpha$ or $\xi R \alpha \gamma$ or $\xi R \beta \alpha$ or $\xi R \alpha \beta$.

7. $\alpha R \beta \gamma$ and $\alpha \neq \beta \neq \gamma \neq \alpha$* imply the existence of δ such that $\delta R \alpha \gamma$ and $\delta R \beta \alpha$.

8. $\alpha R \beta \gamma$, $\varepsilon R \beta \gamma$, $\alpha R \varepsilon \gamma$, $\alpha R \beta \varepsilon$, $\varepsilon \neq \alpha, \beta, \gamma$ imply: the existence of ξ such that $\xi \bar{R} \alpha \varepsilon$, $\xi \bar{R} \varepsilon \alpha$, and the existence of δ' such that $\delta' R \beta \gamma$ implies $\delta' R \xi \gamma$ and $\delta' R \beta \xi$, and the existence of δ'' such that $\delta'' R \gamma \beta$ implies $\delta'' R \xi \beta$ and $\delta'' R \gamma \xi$.

II. *Definitions.*

1. ξ is on $\alpha \beta \gamma$ means, $\alpha R \beta \gamma$ and ($\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$).

2. ξ is on $\alpha \beta$ means, $\alpha \neq \beta$ and the existence of γ such that $\gamma R \alpha \beta$ implies $\gamma R \alpha \xi$ or $\gamma R \xi \beta$. (Cf. Fig. 1; shaded portion, including linear boundary through α, β, indicates possible domain of ξ.)

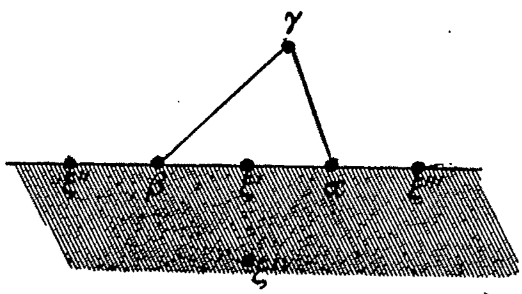


FIG. 1.

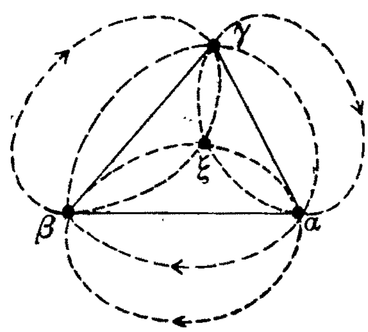


FIG. 2.

3. ξ is in the interior of $\alpha \beta \gamma$ is defined by substituting, in definition 1, "and" for "or". (Cf. Fig. 2.)

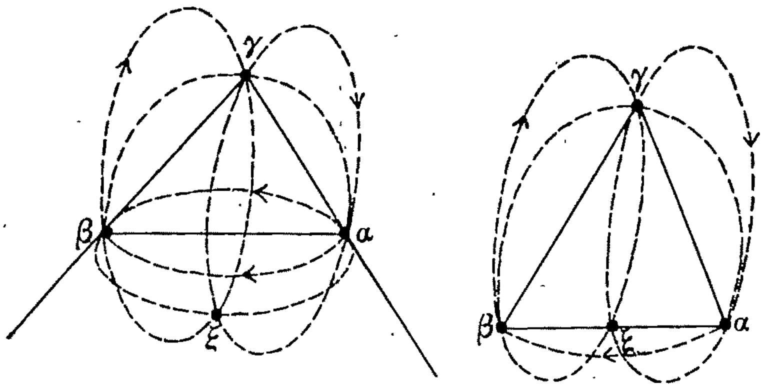
4. ξ is in the interior of $\alpha \beta$ is defined by substituting, in definition 2, "and" for "or". (Cf. Figs. 3.)

* That is, α, β, γ are distinct.

As an equivalent definition under the axioms we may give:

ξ is on $\alpha\beta$ means, $\alpha \neq \beta$ and $\xi \bar{R} \alpha\beta$.

For if $\alpha \neq \beta$ and $\xi \bar{R} \alpha\beta$, the existence of γ such that $\gamma R \alpha\beta$ implies $\gamma R \alpha\xi$ or $\gamma R \xi\beta$, by axiom 5. Conversely, if the existence of γ such that $\gamma R \alpha\beta$ implies $\gamma R \alpha\xi$ or $\gamma R \xi\beta$, then $\xi \bar{R} \alpha\beta$; for $\xi R \alpha\beta$ implies $\xi R \alpha\xi$ or $\xi R \xi\beta$, which in either case is impossible under the axioms.



FIGS. 3.

If we adopt the latter definition, then the corresponding definition of "ξ in the interior of $\alpha\beta$" is:

ξ is in the interior of $\alpha\beta$ means, $\alpha \neq \beta$, $\xi \bar{R} \alpha\beta$ and the existence of γ such that $\gamma R \alpha\beta$ and ($\gamma R \xi\beta$ or $\gamma R \alpha\xi$) implies $\gamma R \xi\beta$ and $\gamma R \alpha\xi$.

On the basis of the above definitions we may define:

ξ is on the plane of the points α, β, γ means, ξ is on $\alpha\beta\gamma$ or $\beta\alpha\gamma$.

ξ is on the line of the points α, β means, ξ is on $\alpha\beta$ and $\beta\alpha$.

ξ is in the interior of, or between,* the points α, β, γ means, ξ is in the interior of $\alpha\beta\gamma$ or $\beta\alpha\gamma$.

ξ is in the interior of, or between, the points α, β means, ξ is in the interior of $\alpha\beta$ and $\beta\alpha$.

III. *Theorems.*

1. $\alpha R \beta\gamma$ implies $\alpha \neq \beta \neq \gamma \neq \alpha$.

By axiom 4, $\alpha R \beta\gamma$ implies $\beta R \gamma\alpha$; similarly, $\beta R \gamma\alpha$ implies $\gamma R \alpha\beta$. Hence, by axiom 3, we have, respectively, $\beta \neq \gamma$, $\gamma \neq \alpha$, $\alpha \neq \beta$; *i. e.*, $\alpha \neq \beta \neq \gamma \neq \alpha$.

* On the terminology cf. Grassmann, "Ausdehnungslehre," 1844, § 110 (Eckgebilde); also Vahlen, "Abstrakte Geometrie," p. 10.

2. There exist three distinct points.

By axioms 1 and 2, and theorem 1.

3. $\xi R \beta \gamma$, $\alpha R \xi \gamma$, $\alpha R \beta \xi$ imply $\alpha R \beta \gamma$.

We have $\xi R \beta \gamma$ and $\alpha \neq \xi, \beta, \gamma$. Hence by axiom 5, $\alpha R \beta \gamma$ or $\xi R \alpha \gamma$ or $\xi R \beta \alpha$. But from the hypothesis it follows that $\xi R \gamma \alpha$ and $\xi R \alpha \beta$. Hence by axiom 3 we must have $\alpha R \beta \gamma$.

4. $\alpha R \beta \gamma$ and $\alpha R \xi \gamma$, $\xi \neq \beta$ imply $\xi R \beta \gamma$ or $\xi R \gamma \beta$ or $\xi R \alpha \beta$ or $\xi R \beta \alpha$.

This follows at once from axiom 6, since we have $\beta R \gamma \alpha$ and $\xi R \gamma \alpha$ by axiom 4.

5. $\xi \bar{R} \alpha \beta$, $\xi \bar{R} \beta \alpha$, $\eta \bar{R} \alpha \beta$, $\eta \bar{R} \beta \alpha$, $\xi \neq \eta$, $\alpha \neq \beta$ imply $\alpha \bar{R} \xi \eta$, $\alpha \bar{R} \eta \xi$, $\beta \bar{R} \xi \eta$, $\beta \bar{R} \eta \xi$.

It is sufficient to prove that $\alpha \bar{R} \xi \eta$, $\alpha \bar{R} \eta \xi$. Suppose $\alpha R \xi \eta$. Then since $\beta \neq \alpha$ and we may assume $\beta \neq \xi, \eta$, we have by axiom 5, $\beta R \xi \eta$ or $\alpha R \beta \eta$ or $\alpha R \xi \beta$. Therefore $\beta R \xi \eta$. But by axiom 6, $\alpha R \xi \eta$ and $\beta R \xi \eta$ imply $\alpha R \beta \eta$ or $\alpha R \eta \beta$ or $\alpha R \xi \beta$ or $\alpha R \beta \xi$. Hence $\beta R \xi \eta$ is impossible; *i. e.*, $\alpha \bar{R} \xi \eta$ is true. Similarly, $\alpha \bar{R} \eta \xi$.

6. $\xi \neq \eta$ implies the existence of ζ such that $\zeta R \xi \eta$.

By axioms 1, 2 there exist the points $\alpha_0, \beta_0, \gamma_0$ such that, say, $\alpha_0 R \beta_0 \gamma_0$. If $\xi \neq \alpha_0$, β_0, γ_0, we have by axiom 5, $\xi R \beta_0 \gamma_0$ or $\alpha_0 R \xi \gamma_0$ or $\alpha_0 R \beta_0 \xi$. Let $\xi R \beta_0 \gamma_0$. Then if $\eta \neq \xi$, β_0, γ_0 by axiom 5, $\eta R \beta_0 \gamma_0$ or $\xi R \eta \gamma_0$ or $\xi R \beta_0 \eta$. If $\eta R \beta_0 \gamma_0$, since $\xi R \beta_0 \gamma_0$ we have by axiom 6, $\xi R \eta \gamma_0$ or $\xi R \gamma_0 \eta$ or $\xi R \beta_0 \eta$ or $\xi R \eta \beta_0$. Suppose $\xi R \gamma_0 \eta$; that is, $\gamma_0 R \eta \xi$. Then by axiom 7 there exists a point ζ such that $\zeta R \xi \eta$.

In the preceding proof is contained implicitly the following theorem:

7. $\alpha R \beta \gamma$, $\xi \neq \eta$ imply $\xi R \eta \alpha$ or $\eta R \xi \alpha$ or $\xi R \eta \beta$ or $\eta R \xi \beta$ or $\xi R \eta \gamma$ or $\eta R \xi \gamma$.

8. $\alpha \neq \beta$ implies the existence of ξ such that ξ is in the interior of α, β.

By theorem 6, $\alpha \neq \beta$ implies the existence of γ such that $\gamma R \alpha \beta$. Then by axiom 7 there is a δ such that $\delta R \gamma \beta$ and $\delta R \alpha \gamma$. Since $\gamma R \alpha \beta$, $\delta R \gamma \beta$ and $\delta R \alpha \gamma$, we have by theorem 3, $\delta R \alpha \beta$. Therefore, by axiom 8, there is a point ξ in the interior of $\alpha \beta$ and $\beta \alpha$.

9. $\alpha R \beta \gamma$, $\alpha' R \gamma \beta$ and $\alpha \neq \alpha'$ imply the (unique) existence of ξ such that $\xi \bar{R} \beta \gamma$, $\xi \bar{R} \gamma \beta$ and ξ is in the interior of α', α.

Since $\beta \neq \gamma$, there exists by theorem 8 a point ξ_0 in the interior of β, γ. Then let, first, $\xi_0 R \alpha \alpha'$ or $\xi_0 R \alpha' \alpha$. Suppose $\xi_0 R \alpha \alpha'$. Since ξ_0 is in the interior of β, γ, $\xi_0 R \gamma \alpha$ and $\xi_0 R \alpha' \gamma$; that is, ξ_0 is in the interior of $\alpha \alpha' \gamma$.

Hence by axiom 8 there exists an η such that η is in the interior of α, α' and $\eta \bar{R} \xi_0 \gamma$, $\eta \bar{R} \gamma \xi_0$. Since $\xi_0 \bar{R} \beta \gamma$ and $\xi_0 \bar{R} \gamma \beta$, it follows by theorem 5 that $\eta \bar{R} \beta \gamma$, $\eta \bar{R} \gamma \beta$. A similar proof holds if $\xi_0 R \alpha' \alpha$.

Let now $\xi_0 \bar{R} \alpha \alpha'$, $\xi_0 \bar{R} \alpha' \alpha$. Since $\xi_0 \neq \gamma$, there is an η in the interior of ξ_0, γ, by theorem 8. Hence $\eta R \gamma \alpha$, $\eta R \alpha \xi$ and $\eta R \alpha' \gamma$ and $\eta R \xi \alpha'$. Since $\eta R \xi \alpha'$ and $\alpha \neq \eta, \xi, \alpha'$ by axiom 5, $\alpha R \xi \alpha'$ or $\eta R \alpha \alpha'$ or $\eta R \xi \alpha$. Hence $\eta R \alpha \alpha'$. That is, η is in the interior of $\alpha \alpha' \gamma$. Hence by axiom 8 there is a point ζ in the interior of α, α' and such that $\zeta \bar{R} \gamma \eta$, $\zeta \bar{R} \eta \gamma$. Since $\eta \bar{R} \xi_0 \gamma$, $\eta \bar{R} \gamma \xi_0$ and $\xi_0 \bar{R} \beta \gamma$, $\xi_0 \bar{R} \gamma \beta$, it follows, by theorem 5, $\zeta \bar{R} \beta \gamma$, $\zeta \bar{R} \gamma \beta$. Thus the theorem is valid.

With the aid of the preceding theorems we can easily derive, on the basis of the axioms 1–8, the usual properties of the (unique) descriptive plane,* excepting, of course, the continuous property. A set of points satisfying system ${}^2R_2$ may be said to be cyclically ordered.

We inquire now into the extensibility of the system ${}^2R_2$. In the extended system, the axiom of dimensionality, axiom 5, will be contradicted, the remaining axioms being valid. There exists a finite system of planar triads such that axioms 1–4, 6–8 are satisfied or are not effective, and axiom 5 is contradicted; it is the system consisting of an arbitrary number of closed chains of the type

$$[\xi \beta \gamma] \qquad [\alpha \xi \gamma] \qquad [\alpha \beta \xi] \qquad [\beta \alpha \gamma].$$

In such a system, axioms 1–4 are satisfied; axiom 5 is contradicted; axiom 6 is not effective; axiom 7 is satisfied; axiom 8 is not effective. The preceding system of triads is, however, not a part of a descriptive geometry; for if so, there would be a point θ such that $\theta R \xi \beta$ and in the plane $\alpha \beta \xi$. Then θ is in the planes $\alpha \beta \xi$, $\xi \beta \gamma$ and α is not in the plane $\xi \beta \gamma$. But this is impossible in a descriptive geometry. Let us require, then, that the extended system be descriptive. Let ξ be not on the plane $\alpha \beta \gamma$. That is, if $\alpha R \beta \gamma$ we have $\xi \bar{R} \beta \gamma$, $\alpha \bar{R} \xi \gamma$, $\alpha \bar{R} \beta \xi$. We may suppose $\alpha R \beta \gamma$, $\xi R_1 \beta \gamma$, $\xi R_2 \gamma \alpha$, $\xi R_3 \alpha \beta$. We consider these cases:

1) $R = R_1$ or R_2 or R_3. If $R = R_1$ ξ is on $\alpha \beta \gamma$, which is contrary to hypothesis. Similarly, $R \neq R_2, R_3$.

2) $\breve{R} = R_1$ or R_2 or R_3, where $\breve{R}$ is the conjugate of R. If $\breve{R} = R_1$, we have $\alpha R \beta \gamma$ and $\xi R \gamma \beta$. Let θ be in the plane $\xi \gamma \beta$ and such that $\theta R \beta \gamma$.

* Cf. E. H. Moore, *Trans. Am. Math. Soc.*, Vol. III (1902), p. 142.

Then θ is in the planes $\alpha\beta\gamma$, $\xi\gamma\beta$, not on the line β, γ, and ξ is not on the plane $\alpha\beta\gamma$. Hence $\breve{R} = R_1$ is impossible; similarly, $\breve{R} = R_2$ or R_3 is impossible.

3) R_1, R_2, $R_3 \neq R$, $\breve{R}$. Suppose $R_1 = R_2$. Then $\xi R_1 \beta\gamma$ and $\xi R_1 \gamma\alpha$, *i. e.*, $\beta R_1 \gamma\xi$, $\alpha R_1 \xi\gamma$. Since ξ is not on the plane $\alpha\beta\gamma$, α is not on the plane $\beta\gamma\xi$. Then we may take a point θ in the plane $\alpha\xi\gamma$ such that $\theta R_1 \gamma\xi$. Then θ is also in the plane $\beta\gamma\xi$. Since θ is not on the line γ, ξ, $R_1 = R_2$ is impossible. Let $\breve{R}_1 = R_2$. Then $\xi R_1 \beta\gamma$ and $\xi R_1 \alpha\gamma$. Hence α is on $\xi\beta\gamma$. That is, ξ is on $\alpha\beta\gamma$. Therefore, $\breve{R}_1 = R_2$ is impossible.

Since the preceding cases are the only essential ones that can arise, we conclude: if ξ is without the plane $\alpha\beta\gamma$, then the corresponding relations

$$R,\ R_1,\ R_2,\ R_3,\ \breve{R},\ \breve{R}_1,\ \breve{R}_2,\ \breve{R}_3$$

are distinct. We may say further, on the basis of the preceding, that if two descriptive planes intersect, the corresponding relations

$$R',\ R'',\ \breve{R}',\ \breve{R}''$$

are distinct. To construct, therefore, an n-dimensional descriptive geometry ($n > 2$) an infinitude of planar relations of the type R is required. However, axioms 1–4, 6–8 are satisfied if we make the agreement that $\alpha R \beta\gamma$ and $\alpha' R \beta'\gamma'$ mean that $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are two euclidean parallel and similarly directed triangular segments; that is, any class of euclidean parallel, similarly directed planes satisfies the above-mentioned axioms; the relation R, then, may be said to be capable of generating in this case an (unlimited) planar vector.

The independence of axioms 1–8 can be established in the following manner. Let us denote by C_n ($n = 1, 2, 3, \ldots, 8$) the class of points such that with respect to this class axiom n is contradicted and the remaining axioms are satisfied or are not effective. Then C_1 consists of no point. C_2 consists of one point α such that $\alpha \bar{R} \alpha\alpha$. C_3 consists of one point α such that $\alpha R \alpha\alpha$. C_4 consists of two points α, β such that $\alpha R \alpha\beta$ and $\alpha \bar{R} \beta\alpha$. C_5 is indicated above. For C_6 we take the ordinary Cartesian plane (rational coordinates) and the point $\iota = (i_1 i_2)$, where ι is distinct from the elements of the preceding plane. The class is ordered thus: the plane is ordered in the usual manner; $\iota R \beta\gamma$ is valid if, and only if, $\omega R \beta\gamma$, where $\omega = (0, 0)$; and if ξ is any point, then $\xi \bar{R} \omega\iota$, $\xi \bar{R} \iota\omega$. The necessary and sufficient condition that $\alpha R \beta\gamma$ is

$$\begin{vmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \\ \gamma_1 & \gamma_2 & 1 \end{vmatrix} > 0,$$

where $\alpha = (\alpha_1 \alpha_2)$, $\beta = (\beta_1 \beta_2)$, $\gamma = (\gamma_1 \gamma_2)$. Thus if $\alpha = \iota$, $\beta_1 \gamma_2 - \beta_2 \gamma_1 > 0$.

C_7 consists of three points α, β, γ with the agreement that $\alpha R \beta \gamma$.

C_8 is the set of points in the usual Cartesian plane with integral coordinates; $\alpha R \beta \gamma$ if, and only if,

$$\begin{vmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \\ \gamma_1 & \gamma_2 & 1 \end{vmatrix} > 0,$$

where $\alpha = (\alpha_1 \alpha_2)$, $\beta = (\beta_1 \beta_2)$, $\gamma = (\gamma_1 \gamma_2)$.

Thus it will be observed that axioms 1–5, 7 are necessary to prove the existence of an infinitude of points.

[We note that for axioms 3, 4, 5 of ${}^2R_2$ the following axioms may be substituted:

3′. $\alpha R \beta \gamma$ implies $\beta \bar{R} \alpha \gamma$.

4′. $\alpha R \beta \gamma$ implies $\gamma \bar{R} \beta \alpha$.

5′. $\alpha R \beta \gamma$, $\xi \neq \alpha$ imply $\xi R \beta \gamma$ or $\xi R \gamma \alpha$ or $\xi R \alpha \beta$.

Theorem. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.

By axiom 3′, $\beta \neq \alpha$; hence if $\alpha R \gamma \beta$, by axiom 5′, $\beta R \gamma \beta$ or $\beta R \beta \alpha$ or $\beta R \alpha \gamma$. By axiom 4′, $\beta R \gamma \beta$ is impossible; $\beta R \beta \alpha$ and $\beta R \alpha \gamma$ are impossible by axiom 3′ and the hypothesis. Hence $\alpha R \gamma \beta$ is impossible.

Theorem. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.

By axiom 3′, $\alpha R \beta \gamma$ implies $\beta \neq \alpha$. Hence by axiom 5′, $\beta R \beta \gamma$ or $\beta R \gamma \alpha$ or $\beta R \alpha \beta$. By axioms 3′ and 4′, $\beta R \beta \gamma$ and $\beta R \alpha \beta$ are impossible; hence $\beta R \gamma \alpha$.

Theorem. $\alpha R \beta \gamma$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.

Proof follows at once from axiom 5′ through the previous theorem.]

System ${}^3R_3$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of $\alpha_0 \beta_0 \gamma_0 \delta_0$ such that $\alpha_0 R \beta_0 \gamma_0 \delta_0$ or $\alpha_0 R \beta_0 \delta_0 \gamma_0$.

3. $\alpha R \beta \gamma \delta$ implies $\alpha \bar{R} \beta \delta \gamma$.

4. $\alpha R \beta \gamma \delta$ implies $\alpha R \gamma \delta \beta$.

5. $\alpha R\beta\gamma\delta$ implies $\gamma R\delta\alpha\beta$.

6. $\alpha R\beta\gamma\delta$, $\xi \neq \alpha,\beta,\gamma,\delta$ imply $\xi R\beta\gamma\delta$ or $\alpha R\xi\gamma\delta$ or $\alpha R\beta\xi\delta$ or $\alpha R\beta\gamma\xi$.

7. $\alpha R\beta\gamma\delta$ and $\xi R\beta\gamma\delta$, $\xi \neq \alpha$ imply $\alpha R\xi\gamma\delta$ or $\alpha R\xi\delta\gamma$ or $\alpha R\beta\xi\delta$ or $\alpha R\beta\delta\xi$ or $\alpha R\beta\gamma\xi$ or $\alpha R\beta\xi\gamma$.

8. $\alpha R\beta\gamma\delta$, $\xi R\beta\gamma\eta$, $\xi \neq \alpha,\delta$, $\eta \neq \alpha,\delta$ imply $\xi R\beta\gamma\delta$ or $\xi R\beta\delta\gamma$ or $\xi R\beta\gamma\alpha$ or $\xi R\beta\alpha\gamma$.

9. $\alpha R\beta\gamma\delta$, $\alpha \neq \beta \neq \gamma \neq \delta \neq \alpha$ imply the existence of ε such that $\varepsilon R\alpha\gamma\delta$, $\varepsilon R\beta\alpha\delta$, $\varepsilon R\beta\gamma\alpha$.

10. $\alpha R\beta\gamma\delta$, $\varepsilon R\beta\gamma\delta$, $\alpha R\varepsilon\gamma\delta$, $\alpha R\beta\varepsilon\delta$, $\alpha R\beta\gamma\varepsilon$, $\varepsilon \neq \alpha,\beta,\gamma,\delta$ imply: the existence of ξ such that $\xi\bar{R}\alpha\beta\varepsilon$, $\xi\bar{R}\alpha\varepsilon\beta$, and the existence of $\delta'\delta''$ such that $\delta' R\delta''\gamma\delta$ implies $\delta' R\delta''\xi\delta$ and $\delta' R\delta''\gamma\xi$.

II. *Definitions.*

1. ξ is on $\alpha\beta\gamma\delta$ means, $\alpha R\beta\gamma\delta$ and ($\xi R\beta\gamma\delta$ or $\alpha R\xi\gamma\delta$ or $\alpha R\beta\xi\delta$ or $\alpha R\beta\gamma\xi$).

2. ξ is on $\alpha\beta\gamma$ means, There exists δ' such that $\delta' R\alpha\beta\gamma$, and the existence of δ such that $\delta R\alpha\beta\gamma$ implies $\delta R\xi\beta\gamma$ or $\delta R\alpha\xi\gamma$ or $\delta R\alpha\beta\xi$.

3. ξ is on $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of $\delta'\delta''$ such that $\delta' R\delta''\alpha\beta$ implies $\delta' R\delta''\xi\beta$ or $\delta' R\delta''\alpha\xi$.

Definitions for ξ in the interior of $\alpha\beta\gamma\delta$, $\alpha\beta\gamma$, $\alpha\beta$ are obtained from the above by substituting for "or", "and". We notice that if ξ is on $\alpha\beta$, ξ is also on $\beta\alpha$; but if ξ is on $\alpha\beta\gamma$, ξ is not necessarily on $\alpha\gamma\beta$.

As equivalent definitions under the axioms we may give:

ξ is on $\alpha\beta\gamma$ means, There exists δ' such that $\delta' R\alpha\beta\gamma$ and $\xi\bar{R}\alpha\beta\gamma$.

ξ is on $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of δ implies $\delta\bar{R}\alpha\beta\xi$.

If we adopt the latter definitions, then we define correspondingly:

ξ is in the interior of $\alpha\beta\gamma$ means, There exists a δ' such that $\delta' R\alpha\beta\gamma$, and the existence of δ such that $\delta R\alpha\beta\gamma$ and ($\delta R\xi\beta\gamma$ or $\delta R\alpha\xi\gamma$ or $\delta R\alpha\beta\xi$) imply $\delta R\xi\beta\gamma$, $\delta R\alpha\xi\gamma$, $\delta R\alpha\beta\xi$.

ξ is in the interior of $\alpha\beta$ means, $\alpha \neq \beta$; the existence of δ implies $\delta\bar{R}\alpha\beta\xi$ and the existence of $\delta'\delta''$ such that $\delta' R\delta''\alpha\beta$ and ($\delta' R\delta''\xi\beta$ or $\delta' R\delta''\alpha\xi$) imply $\delta' R\delta''\xi\beta$, $\delta' R\delta''\alpha\xi$.

On the basis of the above definitions we may define:

ξ is in the space of the points $\alpha,\beta,\gamma,\delta$ means, ξ is on $\alpha\beta\gamma\delta$ or $\alpha\beta\delta\gamma$.

ξ is on the plane of the points α,β,γ means, ξ is on $\alpha\beta\gamma$ and $\alpha\gamma\beta$.

ξ is on the line of the points α,β means, ξ is on $\alpha\beta$.

9. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$, $\xi R \alpha_2\alpha_3\alpha_4\eta_1$, $\xi \neq \alpha_1, \alpha_5$, $\eta_1 \neq \alpha_1, \alpha_5$ imply $\xi R \alpha_2\alpha_3\alpha_4\alpha_5$ or $\xi R \alpha_2\alpha_3\alpha_5\alpha_4$ or $\xi R \alpha_2\alpha_3\alpha_4\alpha_1$ or $\xi R \alpha_2\alpha_3\alpha_1\alpha_4$.

10. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$, $\xi R \alpha_2\alpha_3\eta_1\eta_2$, $\xi \neq \alpha_1, \alpha_4, \alpha_5$, $\eta_1 \neq \alpha_1, \alpha_4, \alpha_5$, $\eta_2 \neq \alpha_1, \alpha_4, \alpha_5$ imply $\xi R \alpha_2\alpha_3\alpha_4\alpha_5$ or $\xi R \alpha_2\alpha_3\alpha_5\alpha_4$ or $\xi R \alpha_2\alpha_3\alpha_5\alpha_1$ or $\xi R \alpha_2\alpha_3\alpha_1\alpha_5$ or $\xi R \alpha_2\alpha_3\alpha_1\alpha_4$ or $\xi R \alpha_2\alpha_3\alpha_4\alpha_1$.

11. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$, $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4 \neq \alpha_5 \neq \alpha_1$ imply the existence of ε such that $\varepsilon R \alpha_1\alpha_3\alpha_4\alpha_5$, $\varepsilon R \alpha_2\alpha_1\alpha_4\alpha_5$, $\varepsilon R \alpha_2\alpha_3\alpha_1\alpha_5$, $\varepsilon R \alpha_2\alpha_3\alpha_4\alpha_1$.

12. $\alpha_1 R \alpha_2\alpha_2\alpha_4\alpha_5$, $\varepsilon R \alpha_2\alpha_3\alpha_4\alpha_5$, $\alpha_1 R \varepsilon\alpha_3\alpha_4\alpha_5$, $\alpha_1 R \alpha_2\varepsilon\alpha_4\alpha_5$, $\alpha_1 R \alpha_2\alpha_3\varepsilon\alpha_5$, $\alpha_1 R \alpha_2\alpha_3\alpha_4\varepsilon$, $\varepsilon \neq \alpha_1$, α_2, α_3, α_4, α_5 imply: the existence of ξ such that $\xi \bar{R} \varepsilon\alpha_3\alpha_4\alpha_5$, $\xi \bar{R} \varepsilon\alpha_3\alpha_5\alpha_4$, and the existence of δ', δ'', δ''' such that $\alpha_1 R \alpha_2\delta'\delta''\delta'''$ imply $\xi R \alpha_2\delta'\delta''\delta'''$, $\alpha_1 R \xi\delta'\delta''\delta'''$.

II. *Definitions.*

1. ξ is on $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ means, $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$ and ($\xi R \alpha_2\alpha_3\alpha_4\alpha_5$ or $\alpha_1 R \xi\alpha_3\alpha_4\alpha_5$ or $\alpha_1 R \alpha_2\xi\alpha_4\alpha_5$ or $\alpha_1 R \alpha_2\alpha_3\xi\alpha_5$ or $\alpha_1 R \alpha_2\alpha_3\alpha_4\xi$).

2. ξ is on $\alpha_1\alpha_2\alpha_3\alpha_4$ means, There exists α_5' such that $\alpha_5' R \alpha_1\alpha_2\alpha_3\alpha_4$, and the existence of α_5 such that $\alpha_5 R \alpha_1\alpha_2\alpha_3\alpha_4$ implies $\alpha_5 R \xi\alpha_2\alpha_3\alpha_4$ or $\alpha_5 R \alpha_1\xi\alpha_3\alpha_4$ or $\alpha_5 R \alpha_1\alpha_2\xi\alpha_4$ or $\alpha_5 R \alpha_1\alpha_2\alpha_3\xi$.

3. ξ is on $\alpha_1\alpha_2\alpha_3$ means, There exist α_4', α_5' such that $\alpha_4' R \alpha_5'\alpha_1\alpha_2\alpha_3$, and the existence of α_4, α_5 such that $\alpha_4 R \alpha_5\alpha_1\alpha_2\alpha_3$ imply that $\alpha_4 R \alpha_5\xi\alpha_2\alpha_3$ or $\alpha_4 R \alpha_5\alpha_1\xi\alpha_3$ or $\alpha_4 R \alpha_5\alpha_1\alpha_2\xi$.

4. ξ is on $\alpha_1\alpha_2$ means, $\alpha_1 \neq \alpha_2$ and the existence of α_3, α_4, α_5 such that $\alpha_3 R \alpha_4\alpha_5\alpha_1\alpha_2$ imply that $\alpha_3 R \alpha_4\alpha_5\xi\alpha_2$ or $\alpha_3 R \alpha_4\alpha_5\alpha_1\xi$.

Definitions for ξ in the interior of $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$, $\alpha_1\alpha_2\alpha_3\alpha_4$, $\alpha_1\alpha_2\alpha_3$, $\alpha_1\alpha_2$ are obtained from the above by substituting "and" for "or". It is readily seen that if ξ is on $\alpha_1\alpha_2$, ξ is on $\alpha_2\alpha_1$; if ξ is on $\alpha_1\alpha_2\alpha_3$, ξ is on $\alpha_2\alpha_1\alpha_3$; but if ξ is on $\alpha_1\alpha_2\alpha_3\alpha_4$, ξ is not necessarily on $\alpha_2\alpha_1\alpha_3\alpha_4$.

Analogous to the definitions under ${}^3R_3$, we may give definitions equivalent to 2, 3, 4. Also we may define:

5. ξ is in the 4-space of the points α_1, α_2, α_3, α_4, α_5 means, ξ is on $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ or $\alpha_2\alpha_1\alpha_3\alpha_4\alpha_5$.

6. ξ is in the 3-space of the points α_1, α_2, α_3, α_4 means, ξ is on $\alpha_1\alpha_2\alpha_3\alpha_4$ and $\alpha_2\alpha_1\alpha_3\alpha_4$.

7. ξ is in the 2-space of the points α_1, α_2, α_3 means, ξ is on $\alpha_1\alpha_2\alpha_3$.

8. ξ is in the 1-space of the points α_1, α_2 means, ξ is on $\alpha_1\alpha_2$.

Definitions for ξ in the interior of the points α_1, α_2, α_3, α_4, α_5, etc., are readily obtained.

System ${}^nR_n'$ $(n \geqq 3)$.

A system of axioms for n-dimensional descriptive geometry $(n \geqq 3)$ is easily constructed on the basis of the foregoing systems. The n-dimensional analogues of axioms 1, 2, 3 of system ${}^4R_4$ are immediately evident. Corresponding to axioms 4, 5, 6 of the same system are a group of $n-1$ axioms which are easily given by means of the rows of the following matrix:

$$\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 \ldots . n-1 & n & n+1 \\
2 & 3 & 1 & 4 & 5 \ldots . n-1 & n & n+1 \\
3 & 4 & 1 & 2 & 5 \ldots . n-1 & n & n+1 \\
4 & 5 & 1 & 2 & 3 \ldots . n-1 & n & n+1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n-2 & n-1 & 1 & 2 & 3 \ldots . n-3 & n & n+1 \\
n-1 & n & 1 & 2 & 3 \ldots . n-3 & n-2 & n+1 \\
n & n+1 & 1 & 2 & 3 \ldots . n-3 & n-2 & n-1
\end{array}$$

Thus, from the first and second rows we get,

$$\alpha_1 R \alpha_2 \alpha_3 \ldots . \alpha_{n+1} \text{ implies } \alpha_2 R \alpha_3 \alpha_1 \ldots . \alpha_{n+1};$$

and from the first and third rows we obtain,

$$\alpha_1 R \alpha_2 \alpha_3 \ldots . \alpha_{n+1} \text{ implies } \alpha_3 R \alpha_4 \alpha_1 \alpha_2 \ldots . \alpha_{n+1};$$

and so on. It is clear that the preceding matrix gives us at once a set of generating substitutions for the alternating group on $n+1$ symbols.*

The n-dimensional extensions of axioms 7 and 8 have no difficulty. Corresponding to axioms 9 and 10 is a set of $n-2$ axioms. These are as follows:

1) $\alpha_1 R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \eta_1$ imply $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_1$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_1 \alpha_n$.

2) $\alpha_1 R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \eta_1 \eta_2$ imply $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_1 \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots . \alpha_{n-1} \alpha_n \alpha_1$.

. . . .

* Cf. E. H. Moore, *Proc. Lond. Math. Soc.*, XXVIII, p. 357. Our generators do not satisfy Moore's relations directly; they correspond to the standpoint of D. N. Lehmer, *Bulletin Am. Math. Soc.*, XIII, p. 81. It is very easy to find generators which satisfy immediately Moore's relations.

$n-3)$ $\alpha_1 R \alpha_2 \alpha_3 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \alpha_4 \eta_1 \eta_3 \ldots \eta_{n-3}$ imply $\xi R \alpha_2 \alpha_3 \alpha_4 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \ldots \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_6 \ldots \alpha_n \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_6 \ldots \alpha_n \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_7 \ldots \alpha_{n+1} \alpha_1 \alpha_5$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_7 \ldots \alpha_{n+1} \alpha_5 \alpha_1$ or .... or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1 \alpha_5 \alpha_6 \ldots \alpha_{n-1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1 \alpha_5 \alpha_6 \ldots \alpha_n \alpha_{n-1}$.

$n-2)$ $\alpha_1 R \alpha_2 \alpha_3 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \eta_1 \eta_2 \ldots \eta_{n-2}$ imply $\xi R \alpha_2 \alpha_3 \ldots \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_6 \ldots \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_6 \ldots \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_6 \alpha_7 \ldots \alpha_1 \alpha_4$ or $\xi R \alpha_2 \alpha_3 \alpha_6 \alpha_7 \ldots \alpha_4 \alpha_1$ or .... or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4 \alpha_5 \ldots \alpha_{n-1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4 \alpha_5 \ldots \alpha_n \alpha_{n-1}$.

The n-dimensional extensions of the remaining axioms are readily obtained. We may also easily give definitions analogous to those under system ${}^4R_4$. The independence of the axioms of system ${}^nR_n$ $(n = 2, 3, \ldots)$, and in particular the associated finite systems, we shall discuss elsewhere.*

CHAPTER V.

The Systems ${}^nK_n$ $(n = 1, 2, 3, \ldots)$.

In a former chapter we have given a set of descriptive systems in terms of an alternating relation R. In this chapter we give for $n = 1, 2, 3$ the corresponding systems in terms of the transitive and symmetrical relation K between two $(n+1)$-ads. Each R-system implies the corresponding K-system. Conversely, on the basis of the K-systems the R-systems are definable.† Thus, in order to define $\alpha\, {}^3R_3\, \beta \gamma \delta$ on the basis of system ${}^3K_3$ we need only put $\alpha\, {}^3R_3\, \beta \gamma \delta = \alpha \beta \gamma \delta\, {}^3K_3\, \alpha_0 \beta_0 \gamma_0 \delta_0$.‡ Notwithstanding the definitional equivalence of the K- and R-systems, it should be noted that as descriptive systems they are essentially distinct;§ for example, they are extended to higher dimensions in very different ways, as we have already pointed out in our introduction.‖

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* Cf. the abstract of the author, *Bulletin of the Am. Math. Soc.*, March, 1908, p. 265.

† Compare B. Russell, "The Principles of Mathematics," pp. 166, 235. Russell's insistence on the asymmetry of relations seems to us somewhat strained; compare, for instance, § 225, p. 236, of the work cited.

‡ See the system ${}^3K_3$ in the present chapter.

§ A logical distinction is that between the relative and absolute standpoints. Compare also p. 314 of our paper, *Transactions Am. Math. Soc.*, Vol. X (1909).

‖ Compare also Chapter IV.

In the systems ${}^nK_n$ ($n = 1, 2, 3, \ldots$) we may replace the statement $\alpha_1\alpha_2\ldots\alpha_{n+1}K\beta_1\beta_2\ldots\beta_{n+1}$ by the congruence*

$$\alpha_1\alpha_2\ldots\alpha_{n+1} \equiv \frac{\alpha_1\alpha_2\ldots\alpha_{n+1}}{\beta_1\beta_2\ldots\beta_{n+1}}\beta_1\beta_2\ldots\beta_{n+1}$$

without impairing the validity of the systems.

System ${}^1K_1$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0 such that $\alpha_0\beta_0K$ or $\beta_0\alpha_0K$.

3. $\alpha\beta K$ implies $\beta\alpha K$.
4. $\alpha\beta K$ implies $\alpha\beta\bar{K}\beta\alpha$.
5. $\alpha\beta K$, $\xi \neq \alpha, \beta$ imply $\xi\beta K$ or $\alpha\xi K$.
6. $\alpha\beta K$, $\xi\beta K$, $\xi \neq \alpha$ imply $\alpha\xi K$.
7. $\xi\beta K\alpha\xi$ implies $\xi\beta K\alpha\beta$.
8. $\alpha\beta K\alpha'\beta'$ implies $\alpha\beta K$.
9. $\alpha\beta K\alpha'\beta'$ implies $\alpha'\beta'K$.
10. $\alpha\beta K\xi\eta$, $\xi\eta K\alpha'\beta'$ imply $\alpha\beta K\alpha'\beta'$.
11. $\alpha\beta K$ implies the existence of ξ such that $\alpha\beta K\xi\alpha$.
12. $\alpha\beta K$ implies the existence of ξ such that $\alpha\beta K\xi\beta$ and $\alpha\beta K\alpha\xi$.
13. $\alpha\beta K$ and $\alpha'\beta'K$ imply $\alpha\beta K\alpha'\beta'$ or $\alpha\beta K\beta'\alpha'$.

II. *Definitions.*

1. $\alpha\beta$ and $\alpha'\beta'$ are collinear means, $\alpha\beta K$, $\alpha'\beta'K$, $\alpha\beta K\alpha'\beta'$ or $\alpha\beta K\beta'\alpha'$.
2. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$, $\alpha\beta K\xi\beta$ or $\alpha\beta K\alpha\xi$.
3. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$, $\alpha\beta K\xi\beta$ and $\alpha\beta K\alpha\xi$.†

System ${}^2K_2$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0, γ_0 such that $\alpha_0\beta_0\gamma_0K$ or $\beta_0\alpha_0\gamma_0K$.

3. $\alpha\beta\gamma K$ implies $\beta\alpha\gamma K$.
4. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma\bar{K}\beta\alpha\gamma$.

* Compare Grassmann, *Gesammelte Werke*, I, 1, p. 127. The terms of this congruence are symmetric products; see Chapter VI for further details.

† An extension of our system ${}^1K_1$ to two dimensions will be found in *Transactions Am. Math. Soc.*, Vol. X, p. 309.

5. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma K\beta\gamma\alpha$.

6. $\alpha\beta\gamma K$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi\beta\gamma K$ or $\alpha\xi\gamma K$ or $\alpha\beta\xi K$.

7. $\alpha\beta\gamma K$, $\xi\beta\gamma K$, $\xi \neq \alpha$ imply $\alpha\xi\gamma K$ or $\alpha\beta\xi K$.

8. $\xi\beta\gamma K\alpha\xi\gamma$, $\xi\beta\gamma K\alpha\beta\xi$ imply $\xi\beta\gamma K\alpha\beta\gamma$.

8′. $\xi\beta\gamma K\alpha\xi\gamma$, $\alpha\beta\xi\bar{K}$ imply $\xi\beta\gamma K\alpha\beta\gamma$.

9. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha\beta\gamma K$.

10. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha'\beta'\gamma' K$.

11. $\alpha\beta\gamma K\xi\eta\zeta$, $\xi\eta\zeta K\alpha'\beta'\gamma'$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$.

12. $\alpha\beta\gamma K$ implies the existence of ξ such that $\alpha\beta\gamma K\xi\alpha\gamma$ and $\alpha\beta\gamma K\xi\beta\alpha$.

13. $\alpha\beta\gamma K$, $\xi\beta\gamma K\alpha\xi\gamma$, $\xi\beta\gamma K\alpha\beta\xi$, $\xi\beta\gamma K\alpha\beta\gamma$ imply: the existence of η such that $\eta\xi\gamma\bar{K}$, and the existence of γ' such that $\alpha\beta\gamma' K$ implies $\alpha\beta\gamma' K\eta\beta\gamma'$ and $\alpha\beta\gamma' K\alpha\eta\gamma'$.

14. $\alpha\beta\gamma K$, $\alpha'\beta'\gamma' K$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

II. *Definitions.*

1. $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are coplanar means, $\alpha\beta\gamma K$, $\alpha'\beta'\gamma' K$, $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

2. ξ is in the 2-space $\alpha\beta\gamma$ means, $\alpha\beta\gamma K$, $\alpha\beta\gamma K\xi\beta\gamma$ or $\alpha\beta\gamma K\alpha\xi\gamma$ or $\alpha\beta\gamma K\alpha\beta\xi$.

3. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$ and $\alpha\beta\xi\bar{K}$.

4. ξ is in the interior of the triangle $\alpha\beta\gamma$ means, $\alpha\beta\gamma K$, $\alpha\beta\gamma K\xi\beta\gamma$, $\alpha\beta\gamma K\alpha\xi\gamma$, $\alpha\beta\gamma K\alpha\beta\xi$.

5. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of γ' such that $\alpha\beta\gamma' K$ implies $\alpha\beta\gamma' K\xi\beta\gamma'$, $\alpha\beta\gamma' K\alpha\xi\gamma'$.

System ${}^3K_3$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0, γ_0, δ_0 such that $\alpha_0\beta_0\gamma_0\delta_0 K$ or $\beta_0\alpha_0\gamma_0\delta_0 K$.

3. $\alpha\beta\gamma\delta K$ implies $\beta\alpha\gamma\delta K$.

4. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta\bar{K}\beta\alpha\gamma\delta$.

5. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta K\beta\gamma\alpha\delta$.

6. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta K\gamma\delta\alpha\beta$.

7. $\alpha\beta\gamma\delta K$, $\xi \neq \alpha, \beta, \gamma, \delta$ imply $\xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or

8. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K$, $\xi \neq \alpha$ imply $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$.

9. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\eta K$, $\xi \neq \alpha, \delta$; $\eta \neq \alpha, \delta$ imply $\xi\beta\gamma\delta K$ or $\xi\beta\gamma\alpha K$.

10. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\xi\beta\gamma\delta K\alpha\beta\gamma\xi$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

10′. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi\bar{K}$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

10″. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta\bar{K}$, $\alpha\beta\gamma\xi\bar{K}$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

11. $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ implies $\alpha\beta\gamma\delta K$.

12. $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ implies $\alpha'\beta'\gamma'\delta' K$.

13. $\alpha\beta\gamma\delta K\xi\eta\zeta\tau$, $\xi\eta\zeta\tau K\alpha'\beta'\gamma'\delta'$ imply $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$.

14. $\alpha\beta\gamma\delta K$ implies the existence of ξ such that $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\xi\beta\alpha\delta$, $\alpha\beta\gamma\delta K\xi\beta\gamma\alpha$.

15. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\xi\beta\gamma\delta K\alpha\beta\gamma\xi$, $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$ imply: the existence of η such that $\eta\gamma\xi\delta K$, and the existence of γ', δ' such that $\alpha\beta\gamma'\delta' K$ implies $\alpha\beta\gamma'\delta' K\alpha\eta\gamma'\delta'$ and $\alpha\beta\gamma'\delta' K\eta\beta\gamma'\delta'$.

16. $\alpha\beta\gamma\delta K$, $\alpha'\beta'\gamma'\delta' K$ imply $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ or $\alpha\beta\gamma\delta K\beta'\alpha'\gamma'\delta'$.

II. *Definitions.*

1. $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$ are cospatial means, $\alpha\beta\gamma\delta K$, $\alpha'\beta'\gamma'\delta' K$, $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ or $\alpha\beta\gamma\delta K\beta'\alpha'\gamma'\delta'$.

2. ξ is in the 3-space $\alpha\beta\gamma\delta$ means, $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

3. ξ is in the 2-space $\alpha\beta\gamma$ means, $\alpha\beta\gamma\xi\bar{K}$ and there exists a δ such that $\alpha\beta\gamma\delta K$.

4. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of δ' implies $\alpha\beta\xi\delta'\bar{K}$.

5. ξ is in the interior of the tetrahedron $\alpha\beta\gamma\delta$ means, $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$, $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$; that is, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta K\alpha\xi\gamma\delta K\alpha\beta\xi\delta K\alpha\beta\gamma\xi$, where the order of the terms of the latter statement is, under the axioms, immaterial.

6. ξ is in the interior of the triangle $\alpha\beta\gamma$ means, There exists δ_1 such that $\alpha\beta\gamma\delta_1 K$; and the existence of δ' such that $\alpha\beta\gamma\delta' K$ implies $\alpha\beta\gamma\delta' K\xi\beta\gamma\delta'$, $\alpha\beta\gamma\delta' K\alpha\xi\gamma\delta'$, $\alpha\beta\gamma\delta' K\alpha\beta\xi\delta'$, $\alpha\beta\gamma\delta' K\alpha\beta\gamma\xi$; that is, $\alpha\beta\gamma\delta' K\xi\beta\gamma\delta' K\alpha\xi\gamma\delta' K\alpha\beta\xi\delta'$.

7. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of γ', δ' such that $\alpha\beta\gamma'\delta' K$ implies $\alpha\beta\gamma'\delta' K\xi\beta\gamma'\delta'$, $\alpha\beta\gamma'\delta' K\alpha\xi\gamma'\delta'$; that is, $\alpha\beta\gamma'\delta' K\xi\beta\gamma'\delta' K\alpha\xi\gamma'\delta'$.

we have, by theorem 7, $\alpha\beta\xi\eta\bar{K}$, $\alpha\gamma\xi\eta\bar{K}$, $\beta\gamma\xi\eta\bar{K}$. Hence $\alpha\beta\gamma\zeta\bar{K}$; *i. e*,. since $\alpha\beta\gamma\xi K_2$, $\alpha\beta\gamma\zeta K_2$.

Theorem 11. If $\alpha\beta\xi\Delta K_1$ and $\alpha\beta\eta\Delta K_1$ and $\xi\neq\eta$, then $\xi\eta\alpha\Delta K_1$ and $\xi\eta\beta\Delta K_1$.

We may suppose $\xi\neq\alpha,\beta$ and $\eta\neq\alpha,\beta$. Since $\alpha\neq\beta$, there exist, by axioms 1, 2 and theorem 5, two points γ,δ such that $\alpha\beta\gamma\delta K$. Then $\alpha\beta\gamma\xi K_2$, $\alpha\beta\delta\xi K_2$, $\alpha\beta\gamma\eta K_2$, $\alpha\beta\delta\eta K_2$. Since $\alpha\beta\gamma\delta K$, $\xi\neq\beta$, we have, by theorem 5, $\xi\beta\gamma\delta K$ or $\xi\beta\delta\alpha K$ or $\xi\beta\alpha\gamma K$. Hence, $\xi\beta\gamma\delta K$. Similarly, $\eta\beta\gamma\delta K$. From $\alpha\beta\gamma\xi K_2$ and $\alpha\beta\gamma\eta K_2$ we have $\xi\beta\gamma\eta\bar{K}$; *i. e.*, since $\xi\beta\gamma\delta K$, $\xi\beta\gamma\eta K_2$. Similarly, $\xi\beta\delta\eta K_2$. Hence, since $\xi\beta\gamma\delta K$, $\xi\beta\gamma\eta K_2$, $\xi\beta\delta\eta K_2$, we have, by theorem 6, $\xi\eta\beta\Delta K_1$. In a similar manner we show that $\xi\eta\alpha\Delta K_1$.

Theorem 12. If $\alpha\beta\gamma\xi K_2$, $\alpha\beta\gamma\eta K_2$, $\alpha\beta\gamma\zeta K_2$ and $\xi\eta\zeta\zeta K_2$, then $\xi\eta\zeta\alpha K_2$, $\xi\eta\zeta\beta K_2$, $\xi\eta\zeta\gamma K_2$.

It will suffice to prove that $\xi\eta\zeta\alpha K_2$. If $\xi\eta\zeta\alpha\bar{K}$, then $\xi\eta\zeta\alpha K_2$. Suppose now $\xi\eta\zeta\alpha K$. Then from $\alpha\beta\gamma\eta K_2$ and $\alpha\beta\gamma\zeta K_2$, we have, by theorem 7, $\alpha\beta\gamma\zeta\bar{K}$. But $\alpha\eta\zeta\xi K$. Hence $\alpha\eta\zeta\beta K_2$. Similarly, $\alpha\xi\zeta\beta K_2$. Therefore, by theorem 6, $\alpha\beta\zeta\Delta K_1$. Similarly, $\alpha\gamma\zeta\Delta K_1$. Since $\zeta\neq\alpha$, we have then, by theorem 11, $\alpha\beta\gamma\Delta K_1$. Since this contradicts the hypothesis, we must have $\xi\eta\zeta\alpha\bar{K}$; *i. e.*, $\xi\eta\zeta\alpha K_2$.

Theorem 13. If $\alpha\beta\gamma\delta K$ and ξ is any point, then $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

We suppose, first, that $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$. Then if the theorem is not verified, by the application of axiom 16, we must have $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\beta\xi\gamma\delta$, $\alpha\beta\gamma\delta K\beta\alpha\xi\delta$, $\alpha\beta\gamma\delta K\beta\alpha\gamma\xi$. Since the K is symmetrical and transitive, we have $\xi\alpha\gamma\delta K\beta\xi\gamma\delta$, $\xi\alpha\gamma\delta K\beta\alpha\xi\delta$, $\xi\alpha\gamma\delta K\beta\alpha\gamma\xi$. Hence, by axiom 10, $\xi\alpha\gamma\delta K\beta\alpha\gamma\delta$. Since $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, we have $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$, which contradicts axiom 4.

If $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi\bar{K}$, then we proceed as above, but employ axiom 10′ instead of axiom 10. Similarly, the theorem is verified by the use of axiom 10″ if $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta\bar{K}$, $\alpha\beta\gamma\xi\bar{K}$.

Finally, if $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta\bar{K}$, $\alpha\beta\xi\delta\bar{K}$, $\alpha\beta\gamma\xi\bar{K}$, we have $\xi=\alpha$ by axiom 8.

Theorem 14. If $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$, then ξ is in the interior of one of precisely fifteen compartments associated with $\alpha\beta\gamma\delta$.

By theorem 13, we have, since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$. Suppose $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$. That is, by axiom 12, $\xi\beta\gamma\delta K$. Since $\alpha\beta\gamma\delta K$ and $\xi\beta\gamma\delta K$, by axiom 8, $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. Hence, by axiom 16, ($\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$) or ($\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\beta\alpha\xi\delta$) or ($\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$ or $\alpha\beta\gamma\delta K\beta\alpha\gamma\xi$). We have then the following eight possibilities:

(1)	(2)	(3)	(4)
$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$

(5)	(6)	(7)	(8)
$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$

If $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$, we again get eight possibilities, leaving, however, only the following new ones:

(9)	(10)	(11)	(12)
$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$

If $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$, from the corresponding eight possibilities, we get two cases:

(13)	(14)
$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$

Finally, if $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$, we get as the only new case:

(15)
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$

The theorem therefore follows. It is plain that of the fifteen compartments indicated above there is a compartment associated with every vertex, every face, and every edge of $\alpha\gamma\beta\delta$, in addition to $\alpha\beta\gamma\delta$ itself. Thus:

1) ξ is in the interior of $\alpha\beta\gamma\delta$,
2) ξ is in the interior of the compartment at α of $\alpha\beta\gamma\delta$, say C_α,
3) ξ is in the interior of the compartment at $\alpha\beta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta}$,
4) ξ is in the interior of the compartment at $\alpha\gamma$ of $\alpha\beta\gamma\delta$, $C_{\alpha\gamma}$,
5) ξ is in the interior of the compartment at $\alpha\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\delta}$,
6) ξ is in the interior of the compartment at $\alpha\beta\gamma$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta\gamma}$,
7) ξ is in the interior of the compartment at $\alpha\gamma\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\gamma\delta}$,
8) ξ is in the interior of the compartment at $\alpha\beta\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta\delta}$,

and so on.

Theorem 15. If $\beta \neq \gamma$, there exists a ξ' in the interior of $\beta\gamma$.

By theorem 5 and axiom 2, there exist two points γ, δ such that $\alpha\beta\gamma\delta K$. Hence, by axiom 14, there is a ξ such that $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\xi\beta\alpha\delta$, $\alpha\beta\gamma\delta K\xi\beta\gamma\alpha$. Hence, by axiom 10, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$. Therefore, by axiom 15, there is a point η in the interior of $\beta\gamma$ such that $\xi\alpha\delta\eta K_2$.

Theorem 15′. If ξ is in the interior of $\beta\gamma$, ξ is in the interior of $\gamma\beta$ and β is not in the interior of $\xi\gamma$.

Since ξ is in the interior of $\beta\gamma$, by definition 7, for any two points δ_1, δ_2 such that $\beta\gamma\delta_1\delta_2 K$, $\beta\gamma\delta_1\delta_2 K\xi\gamma\delta_1\delta_2 K\beta\xi\delta_1\delta_2$. Hence, by theorem 4, for any two points δ_1, δ_2 such that $\gamma\beta\delta_1\delta_2 K$, $\gamma\beta\delta_1\delta_2 K\gamma\xi\delta_1\delta_2 K\xi\beta\delta_1\delta_2$.

Also β is not in the interior of $\xi\gamma$, for if so, for any two points δ_1', δ_2' such that $\xi\gamma\delta_1'\delta_2' K$, $\xi\gamma\delta_1'\delta_2' K\beta\gamma\delta_1'\delta_2' K\xi\beta\delta_1'\delta_2'$. But $\xi\gamma\delta_1\delta_2 K$, since ξ is in the interior of $\beta\gamma$; hence $\xi\gamma\delta_1\delta_2 K\beta\gamma\delta_1\delta_2 K\xi\beta\delta_1\delta_2$, which is impossible by axioms 4, 11, 12.

Theorem 16. If ξ is in the interior of $\alpha\beta$ and η is in the interior of $\alpha\xi$, then η is in the interior of $\alpha\beta$.

By definition and the hypothesis, we have, for any two points δ_1, δ_2 such that $\alpha\beta\delta_1\delta_2 K$, $\alpha\beta\delta_1\delta_2 K\xi\beta\delta_1\delta_2 K\alpha\xi\delta_1\delta_2$, and for any two points δ_1', δ_2' such that $\alpha\xi\delta_1'\delta_2' K$, $\alpha\xi\delta_1'\delta_2' K\eta\xi\delta_1'\delta_2' K\alpha\eta\delta_1'\delta_2'$. Since $\alpha\xi\delta_1\delta_2 K$, $\alpha\xi\delta_1\delta_2 K\eta\xi\delta_1\delta_2 K\alpha\eta\delta_1\delta_2$. Hence $\eta\xi\delta_1\delta_2 K\xi\beta\delta_1\delta_2$, $\eta\beta\xi\delta_1\bar{K}$, $\eta\beta\xi\delta_2\bar{K}$; therefore, by axiom 10″, $\eta\xi\delta_1\delta_2 K\eta\beta\delta_1\delta_2$. Hence $\alpha\beta\delta_1\delta_2 K\eta\beta\delta_1\delta_2 K\alpha\eta\delta_1\delta_2$, *i. e.*, η is in the interior of $\alpha\beta$.

Theorem 17. If $\alpha\beta\gamma\delta K_2$, ξ is in the interior of $\alpha\beta$ and η is in the interior of $\gamma\xi$, then η is in the interior of $\alpha\beta\gamma$.

Let $\alpha\beta\gamma\delta_1 K$. Then since ξ is in the interior of $\alpha\beta$, we have,

$$\alpha\beta\gamma\delta_1 K\xi\beta\gamma\delta_1 K\alpha\xi\gamma\delta_1. \tag{1}$$

Since $\alpha\xi\gamma\delta_1 K$ and η is in the interior of $\xi\gamma$, we have,

$$\xi\gamma\alpha\delta_1 K\eta\gamma\alpha\delta_1 K\xi\eta\alpha\delta_1. \tag{2}$$

Since $\alpha\xi\eta\delta_1 K$ and $\alpha\beta\xi\Delta K_1$, by theorem 8, $\beta\alpha\eta\delta_1 K$. Therefore, since ξ is in the interior of $\alpha\beta$,

$$\alpha\beta\eta\delta_1 K\xi\beta\eta\delta_1 K\alpha\xi\eta\delta_1. \tag{3}$$

Further, since $\xi\gamma\beta\delta_1 K$ and η is in the interior of $\xi\gamma$,

$$\xi\gamma\beta\delta_1 K\eta\gamma\beta\delta_1 K\xi\eta\beta\delta_1. \tag{4}$$

From (1) and (2), $\alpha\beta\gamma\delta_1 K\alpha\eta\gamma\delta_1$; from (1), (2), (3), $\alpha\beta\gamma\delta_1 K\alpha\beta\eta\delta_1$; from (1), (2), (3), (4), $\alpha\beta\gamma\delta_1 K\eta\beta\gamma\delta_1$. Hence η is in the interior of $\alpha\beta\gamma$.

Theorem 18. If $\alpha\beta\gamma\delta K$, then there is a ξ in the interior of $\alpha\beta\gamma$.

Proof follows at once from theorems 15 and 17.

Theorem 19. If $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$, $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ, and ξ is in the interior of $\beta\gamma\delta$, then ξ is in the interior of $\alpha\alpha'\gamma\delta$.

Since ξ is in the interior of $\beta\gamma\delta$ and $\alpha\beta\gamma\delta K$, $\alpha'\beta\gamma\delta K$, by definition 6, we have,

$$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta K\alpha\beta\xi\delta K\alpha\beta\gamma\xi,$$
$$\alpha'\beta\gamma\delta K\alpha'\xi\gamma\delta K\alpha'\beta\xi\delta K\alpha'\beta\gamma\xi.$$

Since $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$, we have $\alpha\beta\xi\delta K\alpha'\xi\beta\delta$, $\alpha\beta\gamma\xi K\alpha'\gamma\beta\xi$, $\alpha\xi\gamma\delta K\alpha'\gamma\xi\delta$. Also, since $\alpha\beta\gamma\delta K$, by theorem 13, $\alpha\beta\gamma\delta K\alpha'\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\alpha'$.

Now $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$, and hence $\alpha\beta\gamma\delta\overline{K}\alpha'\beta\gamma\delta$; and since $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ, $\alpha\beta\gamma\delta\overline{K}\alpha\beta\alpha'\delta$ and $\alpha\beta\gamma\delta\overline{K}\alpha\beta\gamma\alpha'$. Hence we have $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$. In a similar manner it is shown that $\alpha\beta\xi\delta K$ and α' imply $\alpha\beta\xi\delta K\alpha\alpha'\xi\delta$, since $\alpha\beta\xi\delta K\alpha'\xi\beta\delta$ and $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ; and that $\alpha\beta\gamma\xi K$ and α' imply $\alpha\beta\gamma\xi K\alpha\alpha'\gamma\xi$, since $\alpha\beta\gamma\xi K\alpha'\gamma\beta\xi$ and $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ. Therefore, we have $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\xi\delta K\alpha\alpha'\xi\delta$, $\alpha\beta\gamma\xi K\alpha\alpha'\gamma\xi$. But from the preceding, $\alpha\beta\gamma\delta K\alpha\beta\xi\delta K\alpha\beta\gamma\xi K\alpha\xi\gamma\delta K\alpha'\gamma\xi\delta$. Hence

$$\alpha\alpha'\gamma\delta K\alpha\alpha'\xi\delta K\alpha\alpha'\gamma\xi K\alpha\xi\gamma\delta K\xi\alpha'\gamma\delta;$$

that is, ξ is in the interior of $\alpha\alpha'\gamma\delta$.

Theorem 20. If ξ is in the interior of $\beta\gamma\delta$ and η is in the interior of $\alpha\xi$ and $\alpha\beta\gamma\delta K$, then η is in the interior of $\alpha\beta\gamma\delta$.

Since $\alpha\beta\gamma\delta K$ and ξ is in the interior of $\beta\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta K\alpha\beta\xi\delta K\alpha\beta\gamma\xi$. Since η is in the interior of $\alpha\xi$ and $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$, we have $\alpha\xi\gamma\delta K\alpha\eta\gamma\delta$, $\alpha\beta\xi\delta K\alpha\beta\eta\delta$, $\alpha\beta\gamma\xi K\alpha\beta\gamma\eta$. Hence

$$\alpha\eta\gamma\delta K\alpha\beta\eta\delta K\alpha\beta\gamma\eta K\alpha\beta\gamma\delta.$$

Now we have $\eta\beta\gamma\delta K$; for if $\eta\beta\gamma\delta\overline{K}$, then $\alpha\beta\gamma\delta K$, $\eta\beta\gamma\delta\overline{K}$, $\xi\beta\gamma\delta\overline{K}$ imply $\xi\eta\gamma\delta\overline{K}$, by theorem 7; but this contradicts $\xi\gamma\delta\alpha K\xi\gamma\delta\eta$, since η is in the interior of $\alpha\xi$. Since $\eta\beta\gamma\delta K$, we have, by axiom 16, $\alpha\beta\gamma\delta K\eta\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\eta\gamma\beta\delta$.

We suppose $\alpha\beta\gamma\delta K\eta\gamma\beta\delta$. Then since $\eta\beta\gamma\delta K$ and ξ is in the interior of $\beta\gamma\delta$, we have $\eta\beta\gamma\delta K\eta\xi\gamma\delta K\eta\beta\xi\delta K\eta\beta\gamma\xi$; that is, $\eta\beta\gamma\delta K\eta\xi\gamma\delta$. But from the preceding, $\xi\gamma\delta\alpha K\beta\gamma\delta\alpha$, and therefore, since $\eta\gamma\beta\delta K\alpha\beta\gamma\delta$, $\eta\gamma\xi\delta K\alpha\xi\gamma\delta$. Thus $\xi\eta\alpha\Delta\overline{K}$ for any Δ, $\eta\gamma\xi\delta K\alpha\xi\gamma\delta$; also, by theorem 18, there exists a ζ in the interior of $\gamma\xi\delta$; thus we may apply theorem 19. By the latter theorem, ζ is in the interior of $\alpha\eta\gamma\delta$; hence, by axiom 15, there is a ξ' in the interior of $\alpha\eta$ and such that $\xi'\zeta\gamma\delta\overline{K}$. Let us suppose $\xi\neq\xi'$. Then since $\xi'\eta\alpha\Delta K_1$ and $\xi\eta\alpha\Delta K_1$ and $\xi\neq\xi'$, by theorem 11, we have $\alpha\xi\xi'\Delta K_1$. Since $\alpha\xi\gamma\delta K$, by axiom 7, $\xi'\xi\gamma\delta K$ or $\alpha\xi'\gamma\delta K$ or $\alpha\xi\xi'\delta K$ or $\alpha\xi\gamma\xi' K$; that is, since $\xi\xi'\gamma\delta\overline{K}$,* $\alpha\xi\xi'\delta\overline{K}$, $\alpha\xi\xi'\gamma\overline{K}$, $\alpha\xi'\gamma\delta K$. Then $\alpha\xi'\gamma\delta K$ and $\alpha\xi\gamma\delta K$ imply, by axiom 8, $\xi\xi'\alpha\gamma K$ or $\xi\xi'\alpha\delta K$ or $\xi\xi'\gamma\delta K$, which is impossible. Hence $\xi=\xi'$. That is, ξ is in the interior of $\alpha\eta$; but by hypothesis η is in the interior of $\alpha\xi$. Therefore, by theorem 15′, these statements are contradictory, and hence $\alpha\beta\gamma\delta\overline{K}\eta\gamma\beta\delta$; that is, $\alpha\beta\gamma\delta K\eta\beta\gamma\delta$. Hence we have $\alpha\eta\gamma\delta K\alpha\beta\eta\delta K\alpha\beta\gamma\eta K\alpha\beta\gamma\delta K\eta\beta\gamma\delta$; *i. e.*, η is in the interior of $\alpha\beta\gamma\delta$.

Theorem 21. If $\alpha\beta\gamma\delta K$ and ξ is in the interior of $\alpha\beta\gamma$, then there exists an η in the interior of $\beta\gamma$ such that $\alpha\delta\xi\eta K_2$.

By theorem 15 there exists a ζ in the interior of $\delta\xi$; therefore, by theorem 20, ζ is in the interior of $\alpha\beta\gamma\delta$. Hence, by axiom 15, there exists an η in the interior of $\beta\gamma$ such that $\alpha\delta\zeta\eta K_2$; that is, $\alpha\delta\xi\eta K_2$, by theorem 10.

Theorem 22. If $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$ and $\alpha\alpha'\beta\Delta K_1$, then β is in the interior of $\alpha\alpha'$.

By theorem 18, there is a ξ in the interior of $\beta\gamma\delta$. Hence, by theorem 19, the point ξ is in the interior of $\alpha\alpha'\gamma\delta$. Hence, by axiom 15, there is a point β' in the interior of $\alpha\alpha'$ such that $\beta'\gamma\delta\xi\overline{K}$. Then, as in the proof of theorem 20 (in the case $\xi\neq\xi'$), we show that $\beta=\beta'$.

Theorem 23. If $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, then there is a ξ in the interior of $\alpha\alpha'$ such that $\beta\gamma\delta\xi K_2$.

Suppose, first, that $\alpha'\beta\gamma\delta K$, $\alpha\alpha'\gamma\delta K$, $\alpha\beta\alpha'\delta K$, $\alpha\beta\gamma\alpha' K$. We distinguish the following cases:

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* Since $\xi\gamma\delta\alpha K$, $\zeta\xi\gamma\delta\overline{K}$, $\zeta\xi'\gamma\delta\overline{K}$, by theorem 7, $\xi\xi'\gamma\delta\overline{K}$.

CASE I. α' is in the interior of a compartment at a vertex of $\alpha\beta\gamma\delta$, say β.

Then $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\delta\alpha'$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$. That is, since $\alpha\alpha'\gamma\delta K$ and $\alpha\alpha'\gamma\delta K\beta\alpha'\gamma\delta$, $\alpha\alpha'\gamma\delta K\alpha\beta\gamma\delta$, $\alpha\alpha'\gamma\delta K\alpha\alpha'\beta\delta$, $\alpha\alpha'\gamma\delta K\alpha\alpha'\gamma\beta$, β is in the interior of $\alpha\alpha'\gamma\delta$. Hence, by axiom 15, there exists a ξ in the interior of $\alpha\alpha'$ and such that $\beta\gamma\delta\xi K_2$.

CASE II. α' is in the interior of a compartment at an edge of $\alpha\beta\gamma\delta$, say $\beta\gamma$.

Then $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$. Since $\beta \mp \gamma$, there exists, by theorem 15, a point ϕ in the interior of $\beta\gamma$, and we have, therefore, since $\alpha'\delta\beta\gamma K$, $\alpha'\alpha\beta\gamma K$, $\alpha\delta\beta\gamma K$,

$$\alpha'\delta\beta\gamma K\alpha'\delta\varphi\gamma, \quad \alpha'\alpha\beta\gamma K\alpha'\alpha\varphi\gamma, \quad \alpha\delta\beta\gamma K\alpha\delta\varphi\gamma,$$
$$\alpha'\delta\beta\gamma K\alpha'\delta\beta\varphi, \quad \alpha'\alpha\beta\gamma K\alpha'\alpha\beta\varphi, \quad \alpha\delta\beta\gamma K\alpha\delta\beta\varphi.$$

1) If $\alpha\alpha'\delta\phi K$, then we have $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$ or $\alpha\beta\gamma\delta K\alpha'\alpha\delta\phi$. If $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$, then ϕ is in the interior of $\alpha\alpha'\delta\beta$. For we have $\alpha\alpha'\delta\beta K$, and $\alpha'\beta\delta\gamma K\phi\alpha'\delta\beta$, $\alpha\beta\gamma\delta K\alpha\phi\delta\beta$, $\alpha\beta\alpha'\gamma K\alpha\alpha'\phi\beta$, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$. That is, since $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$ and $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\beta$, we have $\alpha\alpha'\delta\beta K\phi\alpha'\delta\beta$, $\alpha\alpha'\delta\beta K\alpha\phi\delta\beta$, $\alpha\alpha'\delta\beta K\alpha\alpha'\phi\beta$, $\alpha\alpha'\delta\beta K\alpha\alpha'\delta\phi$. Thus ϕ is in the interior of $\alpha\alpha'\delta\beta$, and therefore, by axiom 15, there is a point η in the interior of $\alpha\alpha'$ such that $\beta\delta\phi\eta\overline{K}$; *i. e.*, $\beta\delta\phi\eta K_3$. Since $\beta\delta\phi\gamma K_2$, we have then, by theorem 7, $\beta\delta\gamma\eta\overline{K}$; *i. e.*, $\beta\gamma\delta\eta K_2$.

2) Let $\alpha\alpha'\delta\phi\overline{K}$. Then there is a point θ in the interior of $\phi\beta$. Therefore, θ is in the interior of $\beta\gamma$, by theorem 16. To show that $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$. We have $\alpha'\delta\phi\beta K$, $\alpha\alpha'\phi\beta K$, $\alpha\delta\phi\beta K$. Hence

$$\alpha'\delta\varphi\beta K\alpha'\delta\theta\beta, \quad \alpha'\alpha\varphi\beta K\alpha'\alpha\theta\beta, \quad \alpha\delta\varphi\beta K\alpha\delta\theta\beta,$$
$$\alpha'\delta\varphi\beta K\alpha'\delta\varphi\theta, \quad \alpha'\alpha\varphi\beta K\alpha'\alpha\varphi\theta, \quad \alpha\delta\varphi\beta K\alpha\delta\varphi\theta.$$

Since $\alpha'\delta\phi\theta K$, we have, by theorem 13, $\alpha'\delta\phi\theta K\alpha\delta\phi\theta$ or $\alpha'\delta\phi\theta K\alpha\alpha'\phi\theta$ or $\alpha'\delta\phi\theta K\alpha'\delta\alpha\theta$ or $\alpha'\delta\phi\theta K\alpha'\delta\phi\alpha$.

Now we have $\alpha\delta\phi\theta K\alpha\delta\phi\beta$, $\alpha'\delta\phi\theta K\alpha'\delta\phi\beta$ and $\delta\alpha\phi\beta K\alpha'\delta\phi\beta$; hence $\alpha\delta\phi\theta K\alpha'\delta\theta\phi$. Also $\alpha'\delta\phi\theta K\alpha'\delta\phi\beta$, $\alpha'\alpha\phi\theta K\alpha'\alpha\phi\beta$ and $\alpha'\alpha\phi\beta K\delta\alpha'\phi\beta$; hence $\alpha'\delta\phi\theta K\alpha'\alpha\theta\phi$. Finally, we have $\alpha'\alpha\delta\phi\overline{K}$. Hence, we have $\alpha'\delta\phi\theta K\alpha'\delta\alpha\theta$.

But $\alpha'\delta\phi\beta K\alpha'\delta\phi\theta$ and $\alpha\beta\gamma\delta K\phi\alpha'\delta\beta$. Hence $\alpha\beta\gamma\delta K\alpha'\delta\alpha\theta$; *i. e.*, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$, and we may proceed as under 1).

CASE III. α' is in the interior of the compartment at the face $\beta\gamma\delta$ of $\alpha\beta\gamma\delta$.

From the hypothesis, we have $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$, and $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\gamma\delta K\alpha\beta\gamma\alpha'$. Since $\beta \neq \gamma$, there exists a θ in the interior of $\beta\gamma$, by theorem 15. Then since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\alpha' K$ and $\alpha'\beta\gamma\delta K$, we have

$$\alpha\beta\gamma\delta K\alpha\theta\gamma\delta,\quad \alpha'\gamma\beta\delta K\alpha'\theta\beta\delta,\quad \alpha\beta\gamma\alpha' K\alpha\theta\gamma\alpha',$$
$$\alpha\beta\gamma\delta K\alpha\beta\theta\delta,\quad \alpha'\gamma\beta\delta K\alpha'\gamma\theta\delta,\quad \alpha\beta\gamma\alpha' K\alpha\beta\theta\alpha'.$$

1) If $\alpha\alpha'\delta\theta K$, then $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$ or $\alpha\beta\gamma\delta K\alpha'\alpha\delta\theta$. If $\alpha\beta\gamma\delta K\alpha\theta\alpha'\delta$ then α' is in the interior of the compartment at $\theta\delta$ of $\theta\beta\alpha\delta$. For we have $\alpha\beta\gamma\delta K\alpha\beta\theta\delta$, and hence $\alpha\beta\theta\delta K$. Also since $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$ and $\alpha'\gamma\beta\delta K\alpha'\theta\beta\delta$, we have $\alpha\beta\gamma\delta K\alpha'\theta\beta\delta$. Also $\alpha\beta\gamma\delta K\alpha\theta\alpha'\delta$ and $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$. Finally, since $\alpha\beta\gamma\delta K\alpha\beta\gamma\alpha'$ and $\alpha\beta\gamma\alpha' K\alpha\beta\theta\alpha'$, we have $\alpha\beta\gamma\delta K\alpha\beta\theta\alpha'$. Hence $\alpha\beta\theta\delta K\alpha'\theta\beta\delta$, $\alpha\beta\theta\delta K\alpha\theta\alpha'\delta$, $\alpha\beta\theta\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\theta\delta K\alpha\beta\theta\alpha'$. Thus we may proceed further as under Case II, 1). A similar discussion is valid if $\alpha\beta\gamma\delta K\alpha\theta\delta\alpha'$.

2) Let $\alpha\alpha'\delta\theta\overline{K}$. Then, by theorem 15, there is a θ' in the interior of $\theta\delta$, then, by theorem 17, θ' is in the interior of $\beta\gamma\delta$. Hence, by theorem 21, there is a θ'' in the interior of $\beta\delta$ such that $\gamma\alpha\theta'\theta'' K_2$. If $\gamma\alpha\theta''\alpha' K$, then we proceed as under 1). Suppose $\gamma\alpha\theta''\alpha'\overline{K}$; then since $\alpha\delta\theta\alpha'\overline{K}$, $\alpha\delta\theta\theta'\overline{K}$ and $\alpha\delta\theta\beta K$, we have, by theorem 7, $\alpha'\theta'\alpha\delta\overline{K}$. Similarly, $\gamma\alpha\theta''\alpha'\overline{K}$, $\gamma\alpha\theta''\theta'\overline{K}$, $\gamma\alpha\theta''\beta K$ imply $\alpha'\theta'\alpha\theta''\overline{K}$. Since $\theta'\theta''\delta\alpha K$ and $\alpha'\theta'\alpha\delta\overline{K}$, $\alpha'\theta'\alpha\theta''\overline{K}$, $\alpha\alpha'\theta'\Delta\overline{K}$ for any Δ. Moreover, since θ' is in the interior of $\beta\gamma\delta$, we have $\beta\theta'\delta\alpha K\theta'\beta\delta\alpha'$; and, by theorem 18, there is a ξ in the interior of $\beta\theta'\delta$. Hence, by theorem 19, ξ is in the interior of $\alpha\alpha'\beta\delta$.*

Suppose, secondly, that $\alpha'\beta\gamma\delta\overline{K}$ or $\alpha\alpha'\gamma\delta\overline{K}$ or $\alpha\beta\alpha'\delta\overline{K}$ or $\alpha\beta\gamma\alpha'\overline{K}$. We distinguish two cases:

CASE I. $\alpha'\beta\gamma\delta K$, $\alpha\alpha'\gamma\delta K$, $\alpha\beta\alpha'\delta K$, $\alpha\beta\gamma\alpha'\overline{K}$.

Since $\beta \neq \delta$, there exists a ϕ in the interior of $\beta\delta$. Then since $\alpha'\gamma\beta\delta K$, $\alpha\alpha'\beta\delta K$, $\alpha\gamma\beta\delta K$, we have

$$\alpha'\gamma\beta\delta K\alpha'\gamma\varphi\delta,\quad \alpha\alpha'\beta\delta K\alpha\alpha'\varphi\delta,\quad \alpha\gamma\beta\delta K\alpha\gamma\varphi\delta,$$
$$\alpha'\gamma\beta\delta K\alpha'\gamma\beta\varphi,\quad \alpha\alpha'\beta\delta K\alpha\alpha'\beta\varphi,\quad \alpha\gamma\beta\delta K\alpha\gamma\beta\varphi.$$

Hence $\alpha'\gamma\delta\phi K\alpha\delta\gamma\phi$. Also $\alpha'\delta\gamma\phi K$, $\alpha\delta\alpha'\phi K$, $\alpha\delta\gamma\alpha' K$. Further $\alpha\alpha'\gamma\phi K$. For if $\alpha\alpha'\gamma\phi\overline{K}$, since $\alpha\beta\gamma\alpha'\overline{K}$ and $\alpha\gamma\beta\phi K$, then $\alpha\gamma\alpha'\Delta\overline{K}$ for any Δ, by theorem 6. Hence $\alpha\gamma\alpha'\delta\overline{K}$, which contradicts the hypothesis. Hence $\alpha\alpha'\gamma\phi K$, and this case is reduced to the previous discussion.

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* Since $\alpha\alpha'\theta'\Delta K_1$ and $\beta\theta'\delta\alpha K\theta'\beta\delta\alpha'$, by theorem 22, θ' is in the interior of $\alpha\alpha'$; since $\theta'\beta\gamma\delta\overline{K}$, θ' is the point required by theorem.

CASE II. $\alpha' \beta \gamma \delta K$, $\alpha \alpha' \gamma \delta K$, $\alpha \beta \alpha' \delta \overline{K}$, $\alpha \beta \gamma \alpha' \overline{K}$.

Since $\alpha \beta \gamma \delta K$, $\alpha \beta \alpha' \delta \overline{K}$, $\alpha \beta \gamma \alpha' \overline{K}$, then $\alpha \beta \alpha' \Delta \overline{K}$ for any Δ. By theorem 18, there is a ξ in the interior of $\beta \gamma \delta$. Also, by hypothesis, $\alpha \beta \gamma \delta K \alpha' \gamma \beta \delta$. Then we can apply theorem 19 to show that ξ is in the interior of $\alpha \alpha' \gamma \delta$.

Theorem 24. If $\xi \beta \gamma \delta K \alpha \xi \gamma \delta$, $\xi \beta \gamma \delta K \alpha \beta \xi \delta$ and $\alpha \beta \gamma \xi \overline{K}$, then ξ is in the interior of $\alpha \beta \gamma$.

By hypothesis $\gamma \xi \beta \delta K \alpha \beta \xi \delta$, and hence, by theorem 23, there is an η in the interior of $\alpha \gamma$ such that $\xi \beta \delta \eta \overline{K}$. Therefore, since $\alpha \gamma \xi \delta K$ and $\alpha \beta \gamma \delta K$, we have

$$\eta \gamma \xi \delta K \alpha \eta \xi \delta K \alpha \gamma \xi \delta, \tag{1}$$

$$\eta \gamma \beta \delta K \alpha \eta \beta \delta K \alpha \gamma \beta \delta. \tag{2}$$

From the hypothesis and axiom 10′, we have $\alpha \xi \gamma \delta K \alpha \beta \xi \delta$, and from (1) we have $\alpha \xi \eta \delta K \alpha \xi \gamma \delta$; hence $\eta \alpha \xi \delta K \beta \xi \alpha \delta$. Further, we have from (2), $\alpha \beta \eta \delta K$; also $\beta \eta \xi \delta \overline{K}$, and since $\alpha \beta \gamma \xi K_2$ and $\alpha \gamma \eta \Delta K_1$, we have, by theorem 9, $\alpha \beta \xi \eta \overline{K}$. Hence, by theorem 6, $\beta \xi \eta \Delta \overline{K}$ for any Δ. Since $\eta \alpha \xi \delta K \beta \xi \alpha \delta$ and $\beta \xi \eta \Delta K$, by theorem 22, ξ is in the interior of $\beta \eta$. Since $\alpha \beta \gamma \delta K$ and η is in the interior of $\alpha \gamma$ and ξ is in the interior of $\beta \eta$, ξ is in the interior of $\alpha \beta \gamma$ by theorem 17.

Theorem 25. If $\alpha \beta \gamma \delta K$, then there exists a ξ such that $\alpha \beta \gamma \delta K \beta \alpha \xi \delta$ and $\alpha \gamma \xi \Delta K_1$.

Since $\alpha \beta \gamma \delta K$, by axiom 14, there is a point η such that $\alpha \beta \gamma \delta K \eta \alpha \gamma \delta$, $\alpha \beta \gamma \delta K \eta \beta \alpha \delta$, $\alpha \beta \gamma \delta K \eta \beta \gamma \alpha$. That is, by axiom 10, $\alpha \beta \gamma \delta K \eta \beta \gamma \delta$. Since $\alpha \beta \gamma \delta K \eta \beta \gamma \alpha$, by theorem 23, there is an η' in the interior of $\eta \delta$ such that $\alpha \beta \gamma \eta' \overline{K}$.* Since $\eta \delta \beta \alpha K$, $\eta \delta \gamma \beta K$, $\eta \delta \alpha \gamma K$, we have then $\eta \delta \beta \alpha K \eta' \delta \beta \alpha$, $\eta \delta \gamma \beta K \eta' \delta \gamma \beta$, $\eta \delta \alpha \gamma K \eta' \delta \alpha \gamma$. Since $\alpha \beta \gamma \delta K \eta \alpha \gamma \delta$ and $\eta \delta \alpha \gamma K \eta' \delta \alpha \gamma$, we have $\alpha \beta \gamma \delta K \eta' \delta \alpha \gamma$; that is, $\beta \gamma \alpha \delta K \eta' \alpha \gamma \delta$. Hence, by theorem 23, there is a ξ in the interior of $\eta' \beta$ such that $\alpha \gamma \delta \xi \overline{K}$. Now since $\eta' \beta \delta \alpha K$, we have $\eta' \beta \delta \alpha K \xi \beta \delta \alpha$; also $\eta \delta \beta \alpha K \eta' \delta \beta \alpha$ and $\alpha \beta \gamma \delta K \eta \beta \alpha \delta$. Hence $\beta \alpha \gamma \delta K \xi \beta \delta \alpha$; *i. e.*, $\alpha \beta \gamma \delta K \beta \alpha \xi \delta$. Moreover, we have $\eta' \beta \xi \Delta K_1$ and $\alpha \beta \gamma \eta' K_2$. Hence $\eta' \beta \xi \alpha \overline{K}$, $\eta' \beta \alpha \xi K_2$ and $\eta' \beta \alpha \gamma K_2$; therefore, by theorem 7, $\alpha \beta \gamma \xi \overline{K}$; *i. e.*, $\alpha \beta \gamma \xi K_2$. Since $\alpha \beta \gamma \xi K_2$, $\alpha \gamma \delta \xi K_2$ and $\alpha \beta \gamma \delta K$, we have, by theorem 6, $\alpha \gamma \xi \Delta K_1$.

Theorem 26. If α, β are distinct, then there is a ξ such that β is in the interior of $\alpha \xi$.

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* This statement follows also by the direct application of axiom 15, since α is in the interior of $\eta \beta \gamma \delta$.

By theorem 5 and axiom 2, there are two points γ, δ such that $\alpha\beta\gamma\delta K$. By theorem 25, there is a ξ such that $\alpha\beta\xi\Delta K_1$ and $\alpha\beta\gamma\delta K\xi\gamma\beta\delta$. Hence. by theorem 22, β is in the interior of $\alpha\xi$.

Theorem 27. If $\alpha\beta\xi\Delta K_1$ and α, β, ξ are distinct, then one of the points α, β, ξ is in the interior of the other two points.

Since $\alpha \neq \beta$, there exist two points γ, δ such that $\alpha\beta\gamma\delta K$. Then since $\alpha\beta\xi\Delta K_1$ and $\xi \neq \alpha, \beta$, we have $\gamma\alpha\xi\delta K$, $\gamma\beta\xi\delta K$, by theorem 8. Then, by axiom 16, we consider these cases:

$$\beta\gamma\xi\delta K\beta\gamma\alpha\delta, \quad \gamma\alpha\xi\delta K\gamma\alpha\beta\delta. \tag{1}$$

By theorem 22, ξ is in the interior of $\alpha\beta$.

$$\beta\gamma\xi\delta K\beta\gamma\alpha\delta, \quad \gamma\alpha\xi\delta K\alpha\gamma\beta\delta. \tag{2}$$

Then α is in the interior of $\xi\beta$.

$$\beta\gamma\xi\delta K\gamma\beta\alpha\delta, \quad \gamma\alpha\xi\delta K\gamma\alpha\beta\delta. \tag{3}$$

Then β is in the interior of $\xi\alpha$.

$$\beta\gamma\xi\delta K\gamma\beta\alpha\delta, \quad \gamma\alpha\xi\delta K\alpha\gamma\beta\delta. \tag{4}$$

Hence $\beta\gamma\xi\delta K\gamma\alpha\xi\delta$; that is, $\beta\delta\gamma\xi K\alpha\gamma\delta\xi$. Therefore, by theorem 22, ξ is in the interior of $\alpha\beta$. That is, since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta K\alpha\xi\gamma\delta$. Since $\gamma\alpha\xi\delta K\alpha\beta\gamma\delta$ and $\gamma\alpha\xi\delta K\beta\alpha\gamma\delta$, we have $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$, which contradicts axiom 4. Thus case (4) is impossible.

Theorem 28. If $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, then there is no point ξ in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\xi\bar{K}$.

Let a point ξ be in the interior of $\delta\delta'$, supposing that $\delta \neq \delta'$. Then since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta' K$ and $\delta \neq \delta'$, we have, by axiom 8, $\delta'\beta\gamma\delta K$ or $\alpha\delta'\gamma\delta K$ or $\alpha\beta\delta'\delta K$. Let $\alpha\delta'\gamma\delta K$. By theorem 25, there exists a δ'' such that $\delta\gamma\delta''\Delta K_1$ and $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta''$, and by theorem 22, γ is in the interior of $\delta\delta''$. Since $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta''$ and $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, $\alpha\beta\gamma\delta' K\beta\alpha\gamma\delta''$ and hence there is a point η in the interior of $\delta'\delta''$ such that $\alpha\beta\gamma\eta\bar{K}$, by theorem 23. Now $\delta\delta'\delta''\alpha K$, for if $\delta\delta'\delta''\alpha\bar{K}$, $\delta\delta'\delta''\gamma\bar{K}$, $\alpha\delta'\gamma\delta K$ imply $\delta\delta'\delta''\Delta K_1$; also $\delta\delta''\gamma\Delta K_1$, and hence, by theorem 11, $\delta\delta'\gamma\Delta K_1$, which contradicts $\delta\delta'\gamma\alpha K$,

To prove the above theorem, we show then that if $\delta\delta'\delta''\alpha K$, γ is in the interior of $\delta\delta''$, η is in the interior of $\delta'\delta''$, and ξ is in the interior of $\delta\delta'$, then $\alpha\gamma\eta\xi K$. Suppose $\alpha\gamma\eta\xi\bar{K}$; then since $\alpha\delta'\delta''\delta K\alpha\delta'\gamma\delta K\alpha\delta'\delta''\gamma$, we have $\alpha\gamma\delta'\delta'' K$, and hence $\alpha\gamma\delta'\delta'' K\alpha\gamma\eta\delta'' K\alpha\gamma\delta'\eta$; *i. e.*, $\alpha\gamma\delta'\eta K$. Also, since $\delta\delta'\delta''\xi\bar{K}$ and γ is in the interior of $\delta\delta''$, we have $\delta'\delta''\gamma\xi\bar{K}$; and since

$\delta''\delta'\gamma\alpha K$, η is in the interior of $\delta'\delta''$ and $\delta'\delta''\gamma\xi\overline{K}$, we have $\delta'\gamma\eta\xi\overline{K}$. Thus $\alpha\gamma\delta'\eta K$, $\delta'\gamma\eta\xi\overline{K}$, $\alpha\gamma\eta\xi\overline{K}$; therefore, by theorem 6, $\gamma\eta\xi\Delta K_1$. Hence, by theorem 27, since γ, η, ξ are distinct, ξ is in the interior of $\eta\gamma$, or γ is in the interior of $\eta\xi$, or η is in the interior of $\xi\gamma$. Since $\delta\delta'\delta''\alpha K$ and the ξ, η, γ are symmetrically involved with reference to $\delta\delta'\delta''\alpha$, it will suffice to prove that ξ is not in the interior of $\eta\gamma$. Suppose the contrary. Since $\delta\delta'\delta''\alpha K$ and γ is in the interior of $\delta\delta''$, we have $\delta\delta'\delta''\alpha K\gamma\delta'\delta''\alpha K\delta\delta'\gamma\alpha$. Since $\delta'\gamma\delta''\alpha K$ and η is in the interior of $\delta'\delta''$ and ξ is in the interior of $\gamma\eta$, by theorem 17, ξ is in the interior of $\gamma\delta'\delta''$; therefore, $\gamma\delta'\delta''\alpha K\xi\delta'\delta''\alpha K\gamma\xi\delta''\alpha K\gamma\delta'\xi\alpha$. But since ξ is in the interior of $\delta\delta'$ and $\delta\delta'\gamma\alpha K$, we have $\delta\delta'\gamma\alpha K\xi\delta'\gamma\alpha K\delta\xi\gamma\alpha$. Hence $\delta\delta'\gamma\alpha K\xi\delta'\gamma\alpha$ and $\gamma\delta'\delta''\alpha K\gamma\delta'\xi\alpha$. Therefore $\gamma\delta'\delta''\alpha\overline{K}\delta\delta'\gamma\alpha$. The remaining cases are proved in a similar manner. Therefore theorem 27 is contradicted. Hence $\alpha\gamma\xi\eta K$.

Since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\eta\overline{K}$, $\alpha\gamma\xi\eta K$, then $\alpha\beta\gamma\xi K$. For if $\alpha\beta\gamma\xi\overline{K}$, since $\alpha\gamma\xi\eta K$ and $\alpha\beta\gamma\eta\overline{K}$, then $\alpha\beta\gamma\Delta\overline{K}$ for any Δ, which contradicts $\alpha\beta\gamma\delta K$. Thus if ξ is in the interior of $\delta\delta'$ and $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, then $\alpha\beta\gamma\xi K$, which establishes our theorem.

Theorem 29. If $\alpha\beta\gamma\xi K_2$ and $\delta\xi\delta'\xi K_2$, there exists an $\eta \neq \xi$ such that $\alpha\beta\gamma\eta K_2$ and $\delta\xi\delta'\eta K_2$.

If $\alpha\beta\gamma\delta K_2$, then since $\delta\xi\delta'\delta K_2$ and $\delta \neq \xi$, δ is the point required. If $\alpha\beta\gamma\delta' K_2$, then since $\delta\xi\delta'\delta K_2$ and $\delta' \neq \xi$, δ' is the point required. Let $\alpha\beta\gamma\delta K$ and $\alpha\beta\gamma\delta' K$. Then we distinguish the following cases:

CASE I. $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta'$.

Then by theorem 23 there is a point η in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\eta K_2$. Since $\delta'\delta\eta\Delta K_1$ and $\delta\xi\delta'\xi K_2$, we have $\delta\delta'\eta\xi\overline{K}$ and $\delta\xi\delta'\eta K_2$. Also $\xi \neq \eta$; for if $\xi = \eta$, then $\delta\delta'\xi\Delta K_1$, which contradicts $\delta\delta'\xi\xi K_2$.

CASE II. $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$.

Then there is a δ'' such that $\xi\delta\delta''\Delta K_1$ and $\alpha\beta\xi\delta K\beta\alpha\xi\delta''$, by theorem 25, if we suppose $\alpha\beta\xi\delta K$. The latter assumption is permissible, since $\alpha\beta\gamma\delta K$, $\xi \neq \alpha$ or β; if $\xi \neq \alpha$, then, by theorem 5, $\xi\alpha\beta\gamma K$ or $\xi\alpha\beta\delta K$ or $\xi\alpha\gamma\delta K$. Since $\xi\alpha\beta\gamma\overline{K}$, $\xi\alpha\beta\delta K$ or $\xi\alpha\gamma\delta$; we suppose, as indicated above, $\xi\alpha\beta\delta K$. Now $\alpha\beta\xi\delta' K$; for if $\alpha\beta\xi\delta'\overline{K}$, since $\alpha\beta\xi\delta' K_2$ and $\alpha\beta\xi\gamma K_2$, we have, by theorem 7, $\alpha\beta\gamma\delta'\overline{K}$. Also $\alpha\beta\xi\delta' K\alpha\beta\xi\delta$. For otherwise, $\alpha\beta\xi\delta' K\beta\alpha\xi\delta$. Hence, by theorem 23, there is a point η' in the interior of $\delta\delta'$ such that $\alpha\beta\xi\eta' K_2$; that is, since $\alpha\beta\xi\gamma K_2$, by theorem 7, $\alpha\beta\gamma\eta' K_2$. But since $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, there is no η' in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\eta' K_2$, by

heorem 28. Hence $\alpha\beta\xi\delta' K \alpha\beta\xi\delta$. Since, further, $\alpha\beta\xi\delta K \beta\alpha\xi\delta''$, we have $\alpha\beta\xi\delta' K \beta\alpha\xi\delta''$. Therefore, there is a point η in the interior of $\delta'\delta''$ such that $\alpha\beta\xi\eta K_2$. Since $\alpha\beta\xi\eta K_2$ and $\alpha\beta\xi\gamma K_2$, we have, by theorem 7, $\alpha\beta\gamma\eta K_2$. Also $\delta\xi\delta'\xi K_2$ and $\delta\xi\delta''\Delta K_1$; hence $\delta\xi\delta''\delta'\bar{K}$; that is, $\delta\xi\delta'\delta'' K_2$. Since $\delta'\xi\delta\delta'' K_2$ and $\delta'\delta''\eta\Delta K_1$, we have, by theorem 9, $\delta'\xi\delta\eta K_2$. Also $\xi \neq \eta$, for otherwise $\delta\xi\delta''\Delta K_1$ and $\delta''\xi\delta'\Delta K_1$ imply, by theorem 11, $\delta\delta'\xi\Delta K_1$, [illegible] contradicts $\delta\xi\delta'\xi K_2$.

Theorem [illegible] δK, ξ is in the interior of $\alpha\beta$, $\alpha\beta\gamma\eta K_2$, $\xi\eta\alpha\alpha K_2$, $\xi\eta\beta\beta K_2$, $\xi\eta\gamma\gamma K_2$, then there [illegible] ζ such that $\xi\eta\zeta\Delta K_1$, and ζ is in the interior of $\alpha\gamma$ or $\beta\gamma$.

To prove this theorem it will suffice to show that $\xi\eta\delta\alpha K \eta\xi\delta\gamma$ or $\xi\eta\delta\beta K \eta\xi\delta\gamma$. We distinguish two cases.

CASE I. $\alpha\gamma\eta\Delta K_1$ or $\beta\gamma\eta\Delta K_1$.

Then since α, γ, η are distinct, by theorem 27, if $\alpha\gamma\eta\Delta K_1$, γ is in the interior of $\alpha\eta$, or η is in the interior of $\alpha\gamma$, or α is in the interior of $\eta\gamma$. If η is in the interior of $\alpha\gamma$, then we take $\zeta = \eta$. Let γ be in the interior of $\alpha\eta$. Then since $\alpha\eta\beta\delta K$ and $\alpha\eta\xi\delta K$, by theorem 8, and γ is in the interior of $\alpha\eta$,

$$\alpha\eta\beta\delta K \gamma\eta\beta\delta K \alpha\gamma\beta\delta, \tag{1}$$
$$\alpha\eta\xi\delta K \gamma\eta\xi\delta K \alpha\gamma\xi\delta, \tag{2}$$

and since $\alpha\beta\eta\delta K$, $\alpha\beta\gamma\delta K$, and ξ is in the interior of $\alpha\beta$,

$$\alpha\beta\eta\delta K \xi\beta\eta\delta K \alpha\xi\eta\delta, \tag{3}$$
$$\alpha\beta\gamma\delta K \xi\beta\gamma\delta K \alpha\xi\gamma\delta. \tag{4}$$

From (1) and (4), we have $\alpha\eta\beta\delta K \alpha\gamma\xi\delta$; from (2), $\gamma\eta\xi\delta K \alpha\gamma\xi\delta$; from (3), $\alpha\beta\eta\delta K \xi\beta\eta\delta$. Hence $\alpha\eta\beta\delta K \alpha\gamma\xi\delta K \gamma\eta\xi\delta K \xi\eta\beta\delta$. That is, $\gamma\eta\xi\delta K \xi\eta\beta\delta$; *i. e.*, $\eta\xi\delta\gamma K \xi\eta\delta\beta$.

If α is in the interior of $\eta\gamma$, then since $\eta\gamma\beta\delta K$ and ξ is in the interior of $\alpha\beta$, by theorem 17, ξ is in the interior of $\gamma\beta\eta$. Therefore,

$$\gamma\beta\eta\delta K \xi\beta\eta\delta K \gamma\xi\eta\delta K \gamma\beta\xi\delta.$$

Hence $\xi\beta\eta\delta K \gamma\xi\eta\delta$; *i. e.*, $\xi\eta\delta\beta K \eta\xi\delta\gamma$.

CASE II. $\alpha\gamma\eta\eta K_2$ and $\beta\gamma\eta\eta K_2$.

Then, by referring to the proof of theorem 14, it can easily be shown that either (1) η is in the interior of $\alpha\beta\gamma$ or (2) $\alpha\beta\gamma\delta K \eta\alpha\gamma\delta K \alpha\beta\eta\delta$ or (3) $\alpha\beta\gamma\delta K \eta\beta\gamma\delta K \alpha\beta\eta\delta$ or (4) $\alpha\beta\gamma\delta K \beta\eta\gamma\delta K \alpha\eta\gamma\delta$.

If η is in the interior of $\alpha\beta\gamma$, then $\alpha\beta\gamma\delta K \eta\beta\gamma\delta K \alpha\eta\gamma\delta K \alpha\beta\eta\delta$. Since $\alpha\beta\gamma\delta K$ and $\alpha\beta\eta\delta K$ and ξ is in the interior of $\alpha\beta$, we have $\alpha\beta\gamma\delta K \xi\beta\gamma\delta K \alpha\xi\gamma\delta$ and $\alpha\beta\eta\delta K \xi\beta\eta\delta K \alpha\xi\eta\delta$. Since $\xi\beta\gamma\delta K$ and $\xi\gamma\eta\beta\bar{K}$, we have $\xi\gamma\delta\eta K$; for if $\xi\gamma\delta\eta\bar{K}$, then $\xi\gamma\eta\Delta K_1$, which contradicts $\xi\eta\gamma\gamma K_2$. Therefore $\xi\gamma\delta\eta K \alpha\xi\gamma\delta$ or $\xi\gamma\delta\eta K \alpha\gamma\xi\delta$. Suppose $\eta\xi\gamma\delta K \alpha\xi\gamma\delta$. Then we have $\alpha\beta\gamma\delta K \alpha\beta\eta\delta$, $\alpha\beta\eta\delta K \alpha\xi\eta\delta$, $\alpha\beta\gamma\delta K \alpha\xi\gamma\delta$, $\alpha\xi\gamma\delta K \eta\xi\gamma\delta$. Hence $\alpha\xi\eta\delta K \eta\xi\gamma\delta$; that is, $\xi\eta\delta\alpha K \eta\xi\delta\gamma$. Similarly if $\eta\xi\gamma\delta K \alpha$ [illegible]

The remaining sub-cases (2)–(4) are proved in an analogo[illegible]

The preceding descriptive theorems show at once that the relation K, as satisfying the system ${}^3K_3$, suffices to generate projective geometry of three dimensions if one adds to the system ${}^3K_3$ an axiom of continuity. For we may easily identify theorems which we have proved with Hilbert's axioms of connection and order.* Thus our definition of $\alpha\beta\gamma\delta K_1$ corresponds to Hilbert's I, 1; theorem 11 corresponds to I, 2; our definition of $\alpha\beta\gamma\delta K_2$ corresponds to I, 3; theorem 12 corresponds to I, 4; theorem 10 corresponds to I, 5; theorem 29 corresponds to I, 6; theorem 15′ corresponds to II, 1; theorems 15 and 26 correspond to II, 2; theorem 27 corresponds to II, 3; theorem 30 corresponds to II, 5. It is not necessary to prove a theorem corresponding to Hilbert's II, 4, since the latter axiom, as part of Hilbert's system, has been proved redundant.† A direct proof of II, 4 on the basis of our axioms can be very easily given.

* Compare Hilbert's "Foundations of Geometry," translated by E. J. Townsend.

† E. H. Moore, *Transactions American Math. Soc.*, 1902, pp. 142–158, 501; R. L. Moore, *American Math. Monthly*, April, 1902.